

# **DISPERSIVE ESTIMATES FOR DIRAC OPERATORS IN DIMENSION THREE WITH OBSTRUCTIONS AT THRESHOLD ENERGIES**

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ABSTRACT. We investigate  $L^1 \rightarrow L^\infty$  dispersive estimates for the three dimensional Dirac equation with a potential. We also classify the structure of obstructions at the thresholds of the essential spectrum as being composed of a two dimensional space of resonances and finitely many eigenfunctions. We show that, as in the case of the Schrödinger evolution, the presence of a threshold obstruction generically leads to a loss of the natural  $t^{-\frac{3}{2}}$  decay rate. In this case we show that the solution operator is composed of a finite rank operator that decays at the rate  $t^{-\frac{1}{2}}$  plus a term that decays at the rate  $t^{-\frac{3}{2}}$ .

## 1. INTRODUCTION

We consider the linear Dirac equations in three spatial dimensions with potential,

$$(1) \quad i\partial_t \psi(x, t) = (D_m + V(x))\psi(x, t), \quad \psi(x, 0) = \psi_0(x).$$

Here  $x \in \mathbb{R}^3$  and  $\psi(x, t) \in \mathbb{C}^4$ . The  $n$ -dimensional free Dirac operator  $D_m$  is defined by

$$(2) \quad D_m = -i\alpha \cdot \nabla + m\beta = -i \sum_{k=1}^n \alpha_k \partial_k + m\beta,$$

where  $m > 0$  is a constant, and the Hermitian matrices  $\alpha_j$  satisfy

$$(3) \quad \begin{cases} \alpha_j \alpha_k + \alpha_k \alpha_j = 2\delta_{jk} \mathbb{1}_{\mathbb{C}^{2n-1}} & j, k \in \{1, 2, \dots, n\} \\ \alpha_j \beta + \beta \alpha_j = 0_{\mathbb{C}^{2n-1}} \\ \beta^2 = \mathbb{1}_{\mathbb{C}^{2n-1}} \end{cases}$$

Physically,  $m$  represents the mass of the quantum particle. If  $m = 0$  the particle is massless and if  $m > 0$  the particle is massive. We note that dimensions  $n = 2, 3$  are of particular physical importance. In dimension three we use

$$\beta = \begin{bmatrix} I_{\mathbb{C}^2} & 0 \\ 0 & -I_{\mathbb{C}^2} \end{bmatrix}, \quad \alpha_i = \begin{bmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{bmatrix},$$

$$\sigma_1 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

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The Dirac equations (1) were derived by Dirac as an attempt to tie together the theories of relativity and quantum mechanics to describe quantum particles moving at relativistic speeds. The relativistic notion of energy,  $E = \sqrt{c^2 p^2 + m^2 c^4}$ , depends on the particle's mass, momentum and the speed of light. By combining this with the quantum mechanical notions of energy and momentum  $E = i\hbar\partial_t$ ,  $p = -i\hbar\nabla$  one arrives at a non-local equation

$$(4) \quad i\hbar\psi_t(x, t) = \sqrt{-c^2\hbar^2\Delta + m^2c^4} \psi(x, t).$$

We note that this is formally the square root of a Klein-Gordon equation. Dirac's insight was to linearize this equation into a system of four first order equations. This linearization leads to the free Dirac equation, (1) with  $V \equiv 0$ , which describes the evolution of a system of spin up and spin down free electrons and positrons at relativistic speeds. This systemization allows for the study of a first-order evolution equation, in agreement with a quantum mechanical viewpoint. In addition, the linearization allows for the incorporation of external electric or magnetic fields in a relativistically invariant manner, which (4) or a Klein-Gordon equation cannot. Another benefit of this system is to account for the spin of the quantum particles. This interpretation is not without its drawbacks, we refer the reader to the excellent text [37] for a more detailed introduction.

The linearization, (1), retains an important property of (4) in that the free Dirac operator squared generates a diagonal system of Klein-Gordon equations. This motivates the following relationship, which follows from the relationships in (3),

$$(5) \quad (D_m - \lambda)(D_m + \lambda) = (-i\alpha \cdot \nabla + m\beta - \lambda I)(-i\alpha \cdot \nabla + m\beta + \lambda I) = -\Delta + m^2 - \lambda^2.$$

Here the last line is to be interpreted as a diagonal  $4 \times 4$  matrix operator. This allows us to formally define the free Dirac resolvent operator  $\mathcal{R}_0(\lambda) = (D_m - \lambda)^{-1}$  in terms of the free resolvent  $R_0(\lambda) = (-\Delta - \lambda)^{-1}$  of the Schrödinger operator. That is,

$$(6) \quad \mathcal{R}_0(\lambda) = (D_m + \lambda)R_0(\lambda^2 - m^2).$$

Throughout the paper, we use the notation  $X$  to describe a Banach space  $X$  and the Banach spaces of  $C^4$  valued functions with components in  $X$ . Let  $H^1(\mathbb{R}^3)$  be the first order Sobolev space of the  $\mathbb{C}^4$ -valued functions,  $f(x) = (f_i(x))_{i=1}^4$ , of the spatial variable  $x = (x_1, x_2, x_3)$ . Then, the free Dirac operator is essentially self-adjoint on  $H^1(\mathbb{R}^3)$ , its spectrum is purely absolutely continuous and equal to  $\sigma_{ess}(D_m) = \sigma_{ac}(D_m) = (-\infty, -m] \cup [m, \infty)$ , [37, Theorem 1.1]. Under mild assumptions on  $V$ ,  $H := D_m + V$  is self-adjoint, and  $\sigma_{ess}(H) = (-\infty, -m] \cup [m, \infty)$ , [37, Theorem 4.7].

In this paper we aim to study the dispersive bounds by considering the formal solution operator  $e^{-itH}$  as an element of the functional calculus via the Stone's formula:

$$(7) \quad e^{-itH} P_{ac}(H) f(x) = \frac{1}{2\pi i} \int_{(-\infty, -m] \cup [m, \infty)} e^{-it\lambda} [\mathcal{R}_V^+ - \mathcal{R}_V^-](\lambda) f(x) d\lambda,$$

where the perturbed resolvents are defined by  $\mathcal{R}_V^\pm(\lambda) = \lim_{\epsilon \rightarrow 0^+} (D_m + V - (\lambda \pm i\epsilon))^{-1}$ . These resolvent operators are well defined as operators between weighted  $L^2(\mathbb{R}^3)$  spaces, [2], [3]. In particular, in [3, Remark 1.1 and Theorem 3.9], it was shown that this limit is well-defined as an operator from  $H^{0,s}(\mathbb{R}^3)$  to  $H^{1,-s}(\mathbb{R}^3)$  for any  $\lambda \in (-\infty, -m) \cup (m, \infty) \setminus \sigma_p(H)$  and  $s > \frac{1}{2}$  for a class of potentials including those which we consider in Theorems 1.1 and 1.2. Furthermore, for the class of potentials we consider, there are no embedded eigenvalues in the essential spectrum, except possibly at the thresholds  $\lambda = \pm m$ , [38]. See also [34, 7, 39, 23].

It is known that the Dirac operators can have infinitely many eigenvalues in the spectral gap, see for example [37]. However, the work of Cojuhari [13, Theorem 2.1] guarantees only finitely many eigenvalues in the spectral gap for the class of potentials we consider; also see Kurbenin [30].

To discuss our main results, we briefly discuss the notion of threshold resonances and eigenvalues. We characterize both in terms of distributional solutions to the equation

$$H\psi = m\psi.$$

If  $\psi \in L^2(\mathbb{R}^3)$ , we say that there is a threshold eigenvalue at  $\lambda = m$ . If  $\psi \notin L^2(\mathbb{R}^3)$ , but  $\langle x \rangle^{-\frac{1}{2}-\epsilon}\psi \in L^2(\mathbb{R}^3)$  for all  $\epsilon > 0$ , we say that there is a threshold resonance at  $\lambda = m$ . An analogous characterization holds at the threshold  $\lambda = -m$ . We provide a detailed characterization of the threshold in Section 4. If there is neither a threshold resonance or eigenvalue, we say that the threshold is regular.

We take  $\chi \in C_c^\infty(\mathbb{R})$  to be a smooth, even cut-off function of a small neighborhood of the threshold. That is,  $\chi(\lambda) = 1$  if  $|\lambda - m| < \lambda_0$  for a sufficiently small constant  $\lambda_0 > 0$ , and  $\chi(\lambda) = 0$  if  $|\lambda - m| > 2\lambda_0$ . For the duration of the paper, we employ the following notation. We write  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  to indicate that each component of the matrix  $V$  satisfies the bound  $|V_{ij}(x)| \lesssim \langle x \rangle^{-\beta}$ . Our main results are the following low-energy dispersive bounds.

**Theorem 1.1.** *Assume that  $V$  is a Hermitian matrix for which  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  for some  $\beta > 7$ . Further, assume that there is a threshold resonance but not an eigenvalue. Then, there is a time dependent operator  $K_t$ , with rank at most two and satisfying  $\sup_t \|K_t\|_{L^1 \rightarrow L^\infty} \lesssim 1$ , such that*

$$\left\| e^{-itH} P_{ac}(H) \chi(H) - \langle t \rangle^{-\frac{1}{2}} K_t \right\|_{L^1 \rightarrow L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}}.$$

In fact, the operator  $K_t$  in the statement can be written as  $K_t = e^{-imt} P_r + \tilde{K}_t$  where  $P_r$  is a map onto the threshold resonance space (see Proposition 3.5 below) and  $\tilde{K}_t$  is a finite rank operator satisfying the family of weighted bounds  $\|\langle x \rangle^{-j} \tilde{K}_t(x, y) \langle y \rangle^{-j}\|_{L^1 \rightarrow L^\infty} \lesssim \langle t \rangle^{-j}$  for any  $0 \leq j \leq 1$ .

**Theorem 1.2.** *Assume that  $V$  is a Hermitian matrix for which  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  for some  $\beta > 11$ . Further, assume that there is a threshold eigenvalue, then, there is a time dependent, finite rank*

operator  $K_t$  satisfying  $\sup_t \|K_t\|_{L^1 \rightarrow L^\infty} \lesssim 1$ , such that

$$\left\| e^{-itH} P_{ac}(H) \chi(H) - \langle t \rangle^{-\frac{1}{2}} K_t \right\|_{L^1 \rightarrow L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}}.$$

This theorem is valid regardless of the existence or non-existence of threshold resonances. The dynamical, time-decay estimates that we prove provide a valuable contrast to the  $L^2$ -based conservation laws. Using these estimates in concert, one can arrive at many other bounds such as Strichartz estimates for the evolution. Such estimates are often of use when linearizations about special solutions have threshold phenomena for other dispersive equations..

The mathematical analysis of Dirac operators is considerably smaller than the analysis of related equations such as the wave equation, Klein-Gordon or Schrödinger equation. All of the results on three-dimensional Dirac equations in the literature assume that the threshold energies are regular. The first paper that analyzed the time-decay for a perturbed (massless) Dirac equation was [15]. In this paper D’Ancona and Fanelli proved a time-decay rate of  $t^{-1}$  for large  $t$  for the Dirac equation and related magnetic wave equations provided the potential satisfies a certain smallness condition. Escobedo and Vega, [22] provided dispersive and Strichartz estimates for a free Dirac equation in service of analyzing a semi-linear Dirac equation. In [8], Boussaid proved a variety of dispersive estimates for three dimensional Dirac equations. These estimates were in both the weighted  $L^2$  setting and in the sense of Besov spaces. In this paper it was shown that one can obtain faster decay for large  $t$  and smaller singularity as  $t \rightarrow 0$  provided the initial data is smoother in the Besov sense. We rely on the high-energy estimates in [8] to contain our analysis to only a small neighborhood of the threshold. The high-energy portion of the evolution requires smoothness on the initial data and potential, which we do not need for our results. To be precise, by taking  $p = 1$  from Boussaid’s general Besov space result, we see

**Theorem 1.3** ([8], Theorem 1.2). *Assume that  $V$  is a self-adjoint,  $C^\infty$  function that satisfies  $|\partial^\alpha V(x)| \lesssim \langle x \rangle^{\rho+\alpha}$  for some  $\rho > 5$ . Then, for any  $q \in [1, \infty]$ ,  $\theta \in [0, 1]$  with  $s - s' \geq 2 + \theta$ , we have*

$$\left\| e^{-itH} P_{ac}(H) (1 - \chi(H)) \right\|_{B_{1,q}^s \rightarrow B_{\infty,q}^{s'}} \lesssim \begin{cases} |t|^{-1+\frac{\theta}{2}} & 0 < |t| \leq 1 \\ |t|^{-1-\frac{\theta}{2}} & |t| \geq 1 \end{cases}.$$

If we take  $q = 1$ , and  $s' = 0$ , this gives us a  $t^{-\frac{3}{2}}$  decay of the  $L^\infty$  norm of the solution, provided the initial data has two derivatives in  $L^1$  in the Besov sense.

Our approach relies on a detailed analysis of the Dirac resolvent operators. We follow the strategy employed by the first two authors in [19] analyzing the two-dimensional Dirac equation with potential, which has roots in the analysis of the two-dimensional Schrödinger equation by Schlag [35] and the authors [17, 18]. In the same manner we build off the work of the first author and Schlag, [20, 21], in which dispersive estimates for the three-dimensional Schrödinger operators were studied with threshold resonances and/or eigenvalue. These results have been

sharpened, in terms of assumed decay on the potential, by Beceanu [4]. We note that extending these results on the Schrödinger evolution is non-trivial even for the wave equation, see [29].

In addition to proving time decay estimates for the Dirac evolution, we provide a full classification of the obstructions that can occur at the threshold of the essential spectrum at  $\lambda = \pm m$ . For the Schrödinger equation in three dimensions, there can be a one dimensional space of resonances and/or finitely many eigenfunctions at the threshold. This classification is inspired by the previous work on Schrödinger operators [27, 20, 17], though the rich structure of the Dirac operators provides additional technical challenges.

Further study of the Dirac operator in the sense of smoothing and Strichartz estimates has been performed by a variety of authors, see for example [9, 11, 12]. In the two-dimensional case, the evolution on weighted  $L^2$  spaces was studied in [28], which had roots in the work of Murata, [32]. Frequency-localized endpoint Strichartz estimates for the free Dirac equation are obtained in two and three spatial dimensions in [5, 6], which are used to study the cubic non-linear Dirac equation. Dispersive estimates for a one-dimensional Dirac equation were considered in [14].

In the paper we use the following notations. The weighted  $L^2$  space  $L^{2,\sigma}(\mathbb{R}^3) = \{f : \langle \cdot \rangle^\sigma f(\cdot) \in L^2(\mathbb{R}^3)\}$ . We also write  $a- := a - \epsilon$  for an arbitrarily small, but fixed  $\epsilon > 0$ . Similarly,  $a+ := a + \epsilon$ .

The paper is organized as follows. We begin in Section 2 by developing expansions for the Dirac resolvent operators. In Section 3 we prove the dispersive bounds in all cases by reducing the bounds to oscillatory integral estimates. Finally in Section 4 we provide a characterization of the threshold resonances and eigenfunctions.

## 2. RESOLVENT EXPANSIONS AROUND THRESHOLD

In this section we obtain expansions for the resolvent operators  $\mathcal{R}_V^\pm(\lambda)$  in a neighborhood of the threshold energies  $\pm m$ . It is well-known (see e.g. [24]) that the resolvent,  $R_0^\pm(z^2)$ , of the free Schrödinger operator is an integral operator with kernel

$$(8) \quad R_0^\pm(z^2) = \frac{e^{\pm iz|x-y|}}{4\pi|x-y|} = \sum_{j=0}^{\infty} (\pm iz)^j G_j, \quad \text{where}$$

$$(9) \quad G_j(x, y) = \frac{1}{4\pi j!} |x-y|^{j-1} \quad j = 0, 1, 2, \dots,$$

Here we review some estimates (see e.g. [24, 20]) for  $R_0^\pm(z^2)$  needed to study the Dirac evolution. To best utilize these expansions, we employ the notation

$$f(z) = \tilde{O}(g(z))$$

to denote

$$\frac{d^j}{dz^j} f = O\left(\frac{d^j}{dz^j} g\right), \quad j = 0, 1, 2, 3, \dots$$

The notation refers to derivatives with respect to the spectral variable  $z$ , or  $|x - y|$  in the expansions for the integral kernel of the free resolvent operator, which depends on the variable  $\rho = z|x - y|$ . If the derivative bounds hold only for the first  $k$  derivatives we write  $f = \tilde{O}_k(g)$ . In addition, if we write  $f = \tilde{O}_k(1)$ , we mean that differentiation up to order  $k$  is comparable to division by  $z$  and/or  $|x - y|$  as appropriate. This notation applies to operators as well as scalar functions; the meaning should be clear from the context.

In the following analysis we will obtain the expansion on the positive portion  $[m, \infty)$  of the spectrum of  $H$ . A similar analysis with minor changes can be performed to obtain an expansion for the negative portion  $(-\infty, -m]$ , see Remark 2.9.

Writing  $\lambda = \sqrt{m^2 + z^2}$  for  $0 < z \ll 1$ , and using (6), we have

$$(10) \quad \mathcal{R}_0^\pm(\lambda) = [-i\alpha \cdot \nabla + m\beta + \sqrt{m^2 + z^2}I]R_0^\pm(z^2) = \\ [-i\alpha \cdot \nabla + m(\beta + I) + \frac{z^2}{2m}I + O(z^4)I]R_0^\pm(z^2).$$

For convenience we define  $M_{uc}$  and  $M_{lc}$  to be  $4 \times 4$  matrix-valued operators with kernels

$$M_{uc} = \begin{bmatrix} I_{2 \times 2} & 0 \\ 0 & 0 \end{bmatrix}, \quad M_{lc} = \begin{bmatrix} 0 & 0 \\ 0 & I_{2 \times 2} \end{bmatrix}.$$

We also have the following projections  $I_{uc} = \frac{1}{2}(\beta + I)$  and  $I_{lc} = \frac{1}{2}(I - \beta)$ , by

$$I_{uc} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ 0 \\ 0 \end{bmatrix}, \quad I_{lc} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \\ d \end{bmatrix}.$$

In our expansions we will consider only the ‘+’ case due to the simple relationship between the resolvents  $\mathcal{R}_0^\pm(\lambda)$ .

**Lemma 2.1.** *Let  $r := |x - y|$ ,  $\lambda = \sqrt{z^2 + m^2}$ ,  $0 < z < 1$ . We have the following expansions for the free resolvent*

$$(11) \quad \mathcal{R}_0^+(\lambda) = \mathcal{G}_0 + O(z(1 + r^{-1})),$$

$$(12) \quad = \mathcal{G}_0 + iz\mathcal{G}_1 + \tilde{O}_2(z^2r + z^2r^{-1}),$$

$$(13) \quad = \mathcal{G}_0 + iz\mathcal{G}_1 - z^2\mathcal{G}_2 + \tilde{O}_2(z^3r^2 + z^3r^{-1}),$$

$$(14) \quad = \sum_{j=0}^J (iz)^j \mathcal{G}_j + \tilde{O}_2(z^{J+\ell}r^{J+\ell-1} + z^{J+\ell}r^{-1}), \quad J \geq 3,$$

for any  $0 \leq \ell \leq 1$ , where

$$(15) \quad \mathcal{G}_0(x, y) = (D_m + mI)G_0(x, y) = [-i\alpha \cdot \nabla + 2mI_{uc}]G_0(x, y) = \frac{i\alpha \cdot (x - y)}{4\pi|x - y|^3} + \frac{mI_{uc}}{2\pi|x - y|},$$

$$(16) \quad \mathcal{G}_1(x, y) = \frac{m}{2\pi} M_{uc}(x, y),$$

$$(17) \quad \mathcal{G}_2(x, y) = [-i\alpha \cdot \nabla + 2mI_{uc}]G_2(x, y) - \frac{1}{2m}G_0(x, y),$$

$$\mathcal{G}_j(x, y) = O(\langle x - y \rangle^{j-1}), \quad j \geq 3.$$

*Proof.* We will only prove (11) and (14) when  $J = 3$ . The proof of the other expansions and the case  $J > 3$  are similar. First using (8) we have

$$R_0^+(z^2) = \frac{e^{\pm iz|x-y|}}{4\pi|x-y|} = G_0 + O(z), \quad \text{and}$$

$$\nabla R_0^+(z^2) = \nabla G_0 + O(zr^{-1}).$$

The expansion (11) follows immediately.

To obtain (14) when  $J = 3$ , again using (8) we have

$$R_0^+(z^2) = \frac{e^{\pm iz|x-y|}}{4\pi|x-y|} = G_0 + izG_1 - z^2G_2 - iz^3G_3 + \tilde{O}_2(z^{3+\ell}r^{2+\ell}), \quad 0 \leq \ell \leq 1,$$

$$\nabla R_0^+(z^2) = \nabla G_0 - z^2\nabla G_2 - iz^3\nabla G_3 + \tilde{O}_2(z^{3+\ell}r^{1+\ell}), \quad 0 \leq \ell \leq 1.$$

Using this in (10), we have

$$\begin{aligned} \mathcal{R}_0^+(\lambda) &= -i\alpha \cdot [\nabla G_0 - z^2\nabla G_2 - iz^3\nabla G_3] + 2mI_{uc}(G_0 + izG_1 - z^2G_2 - iz^3G_3) \\ &\quad + \frac{z^2}{2m}(G_0 + izG_1) + \tilde{O}_2(z^{3+\ell}r^{1+\ell} + z^{3+\ell}r^{2+\ell} + z^4r^{-1}). \end{aligned}$$

Note that (for  $0 < z < 1$  and  $0 \leq \ell \leq 1$ )

$$\tilde{O}_2(z^{3+\ell}r^{1+\ell} + z^{3+\ell}r^{2+\ell} + z^4r^{-1}) = \tilde{O}_2(z^{3+\ell}r^{2+\ell} + z^{3+\ell}r^{-1}).$$

Collecting the terms with same  $z$  power, and noting that

$$|\nabla G_3| + |G_3| + |G_1| \lesssim \langle x - y \rangle^2$$

yields the claim.  $\square$

To obtain expansions for  $\mathcal{R}_V^\pm(\lambda) = (D_m + V - (\lambda \pm i0))^{-1}$  where  $\lambda = \sqrt{z^2 + m^2}$  we utilize the symmetric resolvent identity. First note that, since  $V : \mathbb{R}^3 \rightarrow \mathbb{C}^{4 \times 4}$  is self-adjoint, we can write

$$V = B^* \Lambda B = B^* |\Lambda|^{\frac{1}{2}} U |\Lambda|^{\frac{1}{2}} B =: v^* U v, \quad \text{where}$$

$$\Lambda = \text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \quad \text{with } \lambda_j \in \mathbb{R},$$

$$|\Lambda|^{\frac{1}{2}} = \text{diag}(|\lambda_1|^{\frac{1}{2}}, |\lambda_2|^{\frac{1}{2}}, |\lambda_3|^{\frac{1}{2}}, |\lambda_4|^{\frac{1}{2}}),$$

$$U = \text{diag}(\text{sign}(\lambda_1), \text{sign}(\lambda_2), \text{sign}(\lambda_3), \text{sign}(\lambda_4)).$$

Defining  $A^\pm(z) = U + v\mathcal{R}_0^\pm(\sqrt{z^2 + m^2})v^*$ , as in [19], the symmetric resolvent identity yields

$$(18) \quad \mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)v^*(A^\pm)^{-1}(z)v\mathcal{R}_0^\pm(\lambda).$$

Note that the statements of Theorems 1.1 and 1.2 control operators  $L^1(\mathbb{R}^3)$  to  $L^\infty(\mathbb{R}^3)$ , while in our analysis we invert  $A^\pm(z)$  in the  $L^2(\mathbb{R}^3)$  setting. Since the leading term of the integral kernel of  $\mathcal{R}_0^\pm(\lambda)$  has size  $|x - y|^{-2}$ , see (15), it does not map  $L^1(\mathbb{R}^3)$  to  $L_{loc}^2(\mathbb{R}^3)$ . However, Remark 2.4 below shows us the iterated resolvents provide a bounded map between these spaces. Therefore to use the symmetric resolvent identity, we need two resolvents on both sides of  $(A^\pm)^{-1}(z)$ . Accordingly we have

$$\mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda) + \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_V^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda).$$

Combining this with (18), we have the identity

$$(19) \quad \mathcal{R}_V^\pm(\lambda) = \mathcal{R}_0^\pm(\lambda) - \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda) + \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda) + \mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda)v^*(A^\pm)^{-1}(z)v\mathcal{R}_0^\pm(\lambda)V\mathcal{R}_0^\pm(\lambda).$$

**Lemma 2.2.** *Let  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  where  $\beta > 2$ , let  $1 \leq l, k < 3$ , with  $l + k < \frac{9}{2}$  and  $\sigma > \frac{1}{2}$ . Then we have*

$$\sup_{x \in \mathbb{R}^3} \left\| \int \frac{1}{|x - x_1|^l} |V(x_1)| \frac{1}{|y - x_1|^k} dx_1 \right\|_{L_y^{2, -\sigma}} \lesssim 1.$$

*The conclusion remains valid in the case  $l$  or  $k$  is zero, provided  $l + k < 3$ ,  $\beta > 3$  and  $\sigma > \frac{3}{2}$ .*

For the proof of Lemma 2.2, we use the following lemma from [16].

**Lemma 2.3.** *Fix  $u_1, u_2 \in \mathbb{R}^n$  and let  $0 \leq k, l < n$ ,  $\beta > 0$ ,  $k + l + \beta \geq n$ ,  $k + l \neq n$ . We have*

$$\int_{\mathbb{R}^n} \frac{\langle x \rangle^{-\beta-}}{|x - u_1|^k |x - u_2|^l} dx \lesssim \begin{cases} \left( \frac{1}{|u_1 - u_2|} \right)^{\max(0, k+l-n)} & |u_1 - u_2| \leq 1, \\ \left( \frac{1}{|u_1 - u_2|} \right)^{\min(k, l, k+l+\beta-n)} & |u_1 - u_2| > 1. \end{cases}$$

*Proof of Lemma 2.2.* Using 2.3 we can obtain the following bound when  $l, k \geq 1$  and  $l + k < \frac{9}{2}$ .

$$\int_{\mathbb{R}^3} \frac{\langle x_1 \rangle^{-\beta-}}{|x - x_1|^k |x_1 - y|^l} dx_1 \lesssim \frac{1}{|x - y|} + \frac{1}{|x - y|^{\frac{3}{2}-}}$$

provided  $\beta > 2$ . Note that when  $k + l = 3$  we can apply the lemma after using the inequality

$$\frac{1}{ab^2} \lesssim \frac{1}{ab^{2-}} + \frac{1}{ab^{2+}} \quad \text{for any } a, b > 0.$$

This yields the first part of the lemma since for  $\sigma > \frac{1}{2}$  we have

$$\sup_{x \in \mathbb{R}^3} \left\| \frac{\langle y \rangle^{-\sigma}}{|x - y|^{\frac{3}{2}-}} \right\|_{L_y^2(\mathbb{R}^3)}, \quad \sup_{x \in \mathbb{R}^3} \left\| \frac{\langle y \rangle^{-\sigma}}{|x - y|} \right\|_{L_y^2(\mathbb{R}^3)} \lesssim 1.$$

If at least one of  $l, k = 0$  then we pick  $\beta > 3$  so that

$$\sup_{x \in \mathbb{R}^3} \int_{\mathbb{R}^n} \frac{\langle x_1 \rangle^{-\beta-}}{|x - x_1|^k |x_1 - y|^l} dx_1 \lesssim 1 \in L_y^{2, -\frac{3}{2}-}(\mathbb{R}^3).$$

□



**Remark 2.4.** Using Lemma 2.2 one can conclude that for any  $|V(x)| \lesssim \langle x \rangle^{-2-}$  and  $\sigma > \frac{1}{2}$ ,

$$\sup_{x \in \mathbb{R}^3} \|\mathcal{R}_0^\pm(\lambda) V \mathcal{R}_0^\pm(\lambda)\|_{L_y^{2,-\sigma}} \lesssim \langle z \rangle^2.$$

Indeed, using (10), we have

$$|\mathcal{R}_0(\lambda)| \lesssim \frac{1}{|x - x_1|^2} + \frac{\langle z \rangle}{|x - x_1|}$$

and accordingly,

$$|\mathcal{R}_0(\lambda)(x, x_1) V(x_1) \mathcal{R}_0(\lambda)(x_1, y)| \lesssim \langle z \rangle^2 \sum_{l, k \in \{1, 2\}} \frac{\langle x_1 \rangle^{-2-}}{|x - x_1|^k |y - x_1|^l}.$$

This gives the claim by Lemma 2.2.

**Definition 2.5.** We say that an operator  $T(z)$  with kernel  $T(x, y)$  is absolutely bounded if  $|T(x, y)|$  gives rise to a bounded operator from  $L^2(\mathbb{R}^3)$  to  $L^2(\mathbb{R}^3)$ . We use the representation  $T(z) = \tilde{O}_j(z^p)$  if  $T(z)$  satisfies the bounds  $\|\partial_z^k T(z)\|_{L^2 \rightarrow L^2} \lesssim z^{p-k}$  for  $k = 1, 2, 3, \dots, j$ .

**Definition 2.6.** An operator  $T$  is Hilbert-Schmidt if its kernel  $T(x, y)$  satisfies

$$\|T\|_{HS}^2 = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |T(x, y)|^2 dx dy < \infty.$$

Hilbert-Schmidt operators and finite rank operators are absolutely bounded.

We have developed expansions for  $\mathcal{R}_0^+(\lambda)$  using the Schrödinger resolvent  $R_0^+(z^2)$ . We develop expansions for  $A(z) := A^+(z)$  when  $z > 0$  and  $A(z) := A^-(-z)$  when  $z < 0$ . It follows from (8) that  $A^-(z) = A^+(-z)$ .

**Lemma 2.7.** Let  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  for some  $\beta > 0$ , and define  $A_0 := U + v\mathcal{G}_0 v^*$ . Then we have the following expansions for  $A(z)$  when  $|z| < 1$ .

$$\begin{aligned} A(z) &= A_0 + izv\mathcal{G}_1 v^* - z^2 v\mathcal{G}_2 v^* - iz^3 v\mathcal{G}_3 v^* + M_0(z), \\ &= A_0 + izv\mathcal{G}_1 v^* - z^2 v\mathcal{G}_2 v^* - iz^3 v\mathcal{G}_3 v^* + z^4 v\mathcal{G}_4 v^* + iz^5 v\mathcal{G}_5 v^* + M_1(z), \quad \text{where} \\ M_0(z) &= \tilde{O}_2(z^{3+}) \text{ if } \beta > 7 \text{ and } M_1(z) = \tilde{O}_2(z^{5+}) \text{ if } \beta > 11. \end{aligned}$$

*Proof.* By the Definition 2.5 it is enough to show that  $\|\partial_z^k M_0(z)(x, y)\|_{HS} \lesssim z^{(3-k)+}$  and  $\|\partial_z^k M_1(z)(x, y)\|_{HS} \lesssim z^{(5-k)+}$  for the given value(s) of  $\beta$ . Using the expansion (14) with  $J = 3$  and  $J = 5$  respectively, and  $\ell = 0+$  we have

$$\begin{aligned} |\partial_z^k M_0(z)(x, y)| &\lesssim z^{(3-k)+} \left( \frac{|v(x)||v^*(y)|}{|x-y|} + |v(x)||x-y|^{2+}|v^*(y)| \right), \\ |\partial_z^k M_1(z)(x, y)| &\lesssim z^{(5-k)+} \left( \frac{|v(x)||v^*(y)|}{|x-y|} + |v(x)||x-y|^{4+}|v^*(y)| \right), \end{aligned}$$

for  $k = 0, 1, 2$ .  $\frac{|v(x)||v^*(y)|}{|x-y|}$  is Hilbert-Schmidt provided  $|v(x)| \lesssim \langle x \rangle^{-1-}$ , and for  $p \geq 0$ ,  $|v(x)||x-y|^p|v^*(y)|$  is Hilbert-Schmidt provided  $|v(x)| \lesssim \langle x \rangle^{-p-\frac{3}{2}-}$ .  $\square$

Lemma 2.7 together with Lemma 2.11 shows that the invertibility of  $A(z)$  as an operator on  $L^2$  for small  $z$  depends upon the invertibility of the operator  $A_0$  on  $L^2$ . Before we discuss the invertibility of  $A(z)$  we give the following definitions for resonances at the threshold  $\lambda = m$ .

- Definition 2.8.** (1) We say that  $\lambda = m$  is a regular point of the spectrum of  $H = D_m + V$  provided  $A_0 = U + v\mathcal{G}_0v^*$  is invertible on  $L^2(\mathbb{R}^3)$ .
- (2) Assume that  $\lambda = m$  is not a regular point of the spectrum. Then we define  $S_1$  as the Riesz projection onto the kernel of  $A_0$  as an operator on  $L^2(\mathbb{R}^3)$ . In this case  $A_0 + S_1$  is invertible. Accordingly we define  $D_0 := (A_0 + S_1)^{-1}$ . We say that there is a resonance of the first kind at the threshold ( $\lambda = m$ ) if  $S_1v\mathcal{G}_1v^*S_1$  is invertible in  $S_1L^2$ , in this case we define  $D_1 := (S_1v\mathcal{G}_1v^*S_1)^{-1}$ .
- (3) Assume  $S_1v\mathcal{G}_1v^*S_1$  is not invertible. Let  $S_2$  be the Riesz projection onto the kernel of  $S_1v\mathcal{G}_1v^*S_1$  as an operator on  $S_1L^2(\mathbb{R}^3)$ . Then  $S_1v\mathcal{G}_1v^*S_1 + S_2$  is invertible on  $S_1L^2(\mathbb{R}^3)$  and we denote  $D_2 := (S_1v\mathcal{G}_1v^*S_1 + S_2)^{-1}$ . We say there is a resonance of the second kind at threshold if  $S_2 = S_1 \neq 0$ . If  $S_2 \neq 0$  and  $S_2 \neq S_1$ , we say there is a resonance of the third kind.

**Remark 2.9.** (i) We provide a full characterization of the threshold obstructions and relate them to various spectral subspaces of  $H = D_m + V$  in Section 4. In particular  $S_1 \neq 0$ ,  $S_1 \neq S_2$  corresponds to the existence of a resonance and  $S_2 \neq 0$  corresponds to the existence of an eigenvalue at the threshold. A resonance of the first kind indicates that there is a threshold resonance but not an eigenvalue.

- (ii) Note that  $v\mathcal{G}_0v^*$  is compact and self-adjoint. Hence,  $A_0$  is a compact perturbation of  $U$  and it is self-adjoint. Also, the spectrum of  $U$  is in  $\{-1, 1\}$ . Hence, zero is the isolated point of the spectrum of  $A_0$  and  $\dim(\text{Ker } A_0)$  is finite. Since  $S_2 \leq S_1$ ,  $S_2$  is also a finite rank projection. In addition, if there is resonance of the first kind then the range of  $S_1$  is at most two dimensional, see Corollary 4.4. Heuristically, the rank of  $S_1$  being at most two corresponds to the possibility of having a ‘spin up’ and a ‘spin down’ resonance function at the threshold energy.
- (iii) We do our analysis in the positive portion of the spectrum  $[m, \infty)$  and develop expansions of  $\mathcal{R}_V$  around the threshold  $\lambda = m$ . One can do the same analysis for the negative portion of the spectrum taking  $\lambda = -\sqrt{z^2 + m^2}$ . In this case the perturbed equation has a threshold resonance or eigenvalue at  $\lambda = -m$  is related to distributional solutions of  $(H + mI)g = 0$ .
- (iv) We have

$$D_0S_1 = S_1D_0 = S_1,$$

and similarly for  $S_2$  and  $D_2$ . We prove below that  $D_0$  is absolutely bounded. The absolute boundedness of  $D_1$ ,  $D_2$  is clear since they are finite rank operators.

**Lemma 2.10.** *The operator  $D_0$  is absolutely bounded in  $L^2(\mathbb{R}^3)$ .*

*Proof.* Recall that  $D_0 = (U + v\mathcal{G}_0v^* + S_1)^{-1}$ . Using the resolvent identity twice we obtain

$$(20) \quad D_0 = U - U(v\mathcal{G}_0v^* + S_1)U + D_0(v\mathcal{G}_0v^* + S_1)U(v\mathcal{G}_0v^* + S_1)U.$$

Note that  $U$  is absolutely bounded. Also note that since  $S_1$  is finite rank, any summand containing  $S_1$  is finite rank, and hence absolutely bounded. Using (15), we have

$$|\mathcal{G}_0(x, y)| \leq c_1 I_1(x, y) + c_2 I_2(x, y),$$

where  $I_1$  and  $I_2$  are the fractional integral operators. One can see that these two operators are compact operators on  $L^{2,\sigma} \rightarrow L^{2,-\sigma}$  for  $\sigma > 1$ , see Lemma 2.3 in [25]. Therefore  $v\mathcal{G}_0v^*$  is absolutely bounded.

It remains to prove that

$$(21) \quad D_0v\mathcal{G}_0v^*Uv\mathcal{G}_0v^*U = D_0v\mathcal{G}_0V\mathcal{G}_0v^*U$$

is absolutely bounded. Recalling the definition of  $\mathcal{G}_0$  given with (15) one can see that the operator  $v\mathcal{G}_0V\mathcal{G}_0v^*U$  is Hilbert-Schmidt by Lemma 2.2 for any  $|v(x)| \lesssim \langle x \rangle^{-2-}$ . Finally, being the composition of a bounded operator,  $D_0$ , and a Hilbert-Schmidt operator,  $v\mathcal{G}_0V\mathcal{G}_0v^*U$ , (21) is Hilbert-Schmidt and hence absolutely bounded.  $\square$

We use the following lemma from [27] to invert the operator  $A(z) = U + v\mathcal{R}_0(\sqrt{z^2 + m^2})v^*$  around  $z = 0$ , ( $\lambda = m$ ).

**Lemma 2.11.** *Let  $\mathbb{F} \subset \mathbb{C} \setminus \{0\}$  have zero as an accumulation point. Let  $A(z)$ ,  $z \in \mathbb{F}$ , be a family of bounded operators of the form*

$$A(z) = A_0 + zA_1(z)$$

*with  $A_1(z)$  uniformly bounded as  $z \rightarrow 0$ . Suppose that  $z = 0$  is an isolated point of the spectrum of  $A_0$ , and let  $S$  be the corresponding Riesz projection. Assume that  $\text{rank}(S) < \infty$ . Then for sufficiently small  $z \in \mathbb{F}$  the operators*

$$(22) \quad B(z) := \frac{1}{z}(S - S(A(z) + S)^{-1}S)$$

*are well-defined and bounded on  $\mathcal{H}$ . Moreover, if  $A_0 = A_0^*$ , then they are uniformly bounded as  $z \rightarrow 0$ . The operator  $A(z)$  has bounded inverse in  $\mathcal{H}$  if and only if  $B(z)$  has a bounded inverse in  $S\mathcal{H}$ , and in this case*

$$(23) \quad A^{-1}(z) = (A(z) + S)^{-1} + \frac{1}{z}(A(z) + S)^{-1}SB^{-1}(z)S(A(z) + S)^{-1}.$$

**Lemma 2.12.** *Suppose that  $\lambda = m$  is not a regular point of the spectrum of  $H = D_m + V$ , with  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  for some  $\beta > 0$ , and let  $S_1$  be the Riesz projection from Definition 2.8. Then*

for sufficiently small  $z_0 > 0$ , the operator  $A(z) + S_1$  is invertible for all  $0 < |z| < z_0 < 1$  as a bounded operator on  $L^2(\mathbb{R}^3) \rightarrow L^2(\mathbb{R}^3)$ . Further, one has

$$(24) \quad \begin{aligned} (A(z) + S_1)^{-1} &= D_0 - iz[D_0 v \mathcal{G}_1 v^* D_0] + z^2[D_0 v \mathcal{G}_2 v^* D_0 - D_0 v \mathcal{G}_1 v^* D_0 v \mathcal{G}_1 v^* D_0] \\ &\quad + z^3 \Gamma_0 + \tilde{O}_3(z^{3+}) \text{ for } \beta > 7, \end{aligned}$$

$$(25) \quad \begin{aligned} (A(z) + S_1)^{-1} &= D_0 - iz[D_0 v \mathcal{G}_1 v^* D_0] + z^2[D_0 v \mathcal{G}_2 v^* D_0 - D_0 v \mathcal{G}_1 v^* D_0 v \mathcal{G}_1 v^* D_0] \\ &\quad + z^3 \Gamma_0 + z^4 \Gamma_1 + z^5 \Gamma_2 + \tilde{O}_5(z^{5+}) \text{ for } \beta > 11. \end{aligned}$$

Here  $\Gamma_0$ ,  $\Gamma_1$  and  $\Gamma_2$  are  $z$  independent absolutely bounded operators.

*Proof.* We use Neumann series expansion using Lemma 2.7. The operators  $\Gamma_0$ ,  $\Gamma_1$  and  $\Gamma_2$  are absolutely bounded since they are composition of Hilbert Schmidt operators with absolutely bounded operators.  $\square$

The following lemma gives an expansion for  $A^{-1}(z)$  for  $0 < |z| < z_0$  when there is a resonance of the first kind at threshold energy.

**Lemma 2.13.** *Let  $|V(x)| \lesssim \langle x \rangle^{-7-}$ . If there is a resonance of the first kind at the threshold  $\lambda = m$ , then*

$$A^{-1}(z) = -\frac{i}{z} S_1 D_1 S_1 + E(z)$$

where  $E(z)$  is an absolutely bounded operator satisfying

$$\left\| \sup_{|z| < z_0} |\partial_z^k E(z)| \right\|_{L^2 \rightarrow L^2} \lesssim 1$$

for  $k = 0, 1$ , and  $\|\partial_z^2 E(z)\|_{L^2 \rightarrow L^2} \lesssim z^{-1+}$ .

*Proof.* Recall that using Lemma 2.11 in order to invert  $A(z)$  first we need to check the invertibility of

$$B(z) = \frac{1}{z} (S_1 - S_1 (A(z) + S_1)^{-1} S_1)$$

on  $S_1 L^2$ . Noting that  $S_1 D_0 = S_1$  and using (24), we have

$$(26) \quad B(z) = i S_1 v \mathcal{G}_1 v^* S_1 - z [S_1 v \mathcal{G}_2 v^* S_1 - S_1 v \mathcal{G}_1 v^* D_0 v \mathcal{G}_1 v^* S_1] + z^2 S_1 \Gamma_0 S_1 + \tilde{O}_2(z^{2+}).$$

Recall by Definition 2.8, if there is a resonance of the first kind then  $S_1 v \mathcal{G}_1 v^* S_1$  is invertible. Hence,  $B(z)$  is invertible and for sufficiently small  $z$ , we have

$$(27) \quad B^{-1}(z) = -i D_1 + z \Gamma_3 + z^2 \Gamma_4 + \tilde{O}_2(z^{2+}).$$

Note that  $\Gamma_i$ 's in here are composition of  $z$  independent, absolutely bounded operators. The absolute boundedness follows since  $S_1$  is finite rank.

Using this expression together with (24) in (23), we have

$$A^{-1}(z) = (A(z) + S_1)^{-1} + \frac{1}{z} (A(z) + S_1)^{-1} S_1 B^{-1}(z) S_1 (A(z) + S_1)^{-1}$$

$$= -\frac{i}{z}S_1D_1S_1 + z\Gamma_5 + \tilde{O}_2(z^{1+}) = -\frac{i}{z}S_1D_1S_1 + E(z).$$

The bounds on the operator  $E(z)$  follow from (24) and (27).  $\square$

The following lemma gives the expansion for  $A^{-1}(z)$  in the cases when there is a resonance of the second or third kind at the threshold, that is when there is a threshold eigenvalue.

**Lemma 2.14.** *Let  $|V(x)| \lesssim \langle x \rangle^{-11-}$ . If there is a resonance of the second or third kind at the threshold  $\lambda = m$ , then we have*

$$A^{-1}(z) = -\frac{1}{z^2}S_2D_3S_2 + \frac{1}{z}\Omega + E(z).$$

where  $S_2D_3S_2$  and  $\Omega$  are finite rank operators. Furthermore,

$$\| \sup_{|z| < z_0} |\partial_z^k E(z)| \|_{L^2 \rightarrow L^2} \lesssim 1, \text{ for } k = 0, 1, \text{ and } \| |\partial_z^2 E(z)| \|_{L^2 \rightarrow L^2} \lesssim z^{-1+}.$$

*Proof.* Recall that in this case the operator  $S_1v\mathcal{G}_1v^*S_1$  is not invertible and we defined  $S_2$  to be the projection on the kernel of  $S_1v\mathcal{G}_1v^*S_1$ . In the following proof we use Lemma 2.11 twice; to first invert  $B(z)$  and then to invert  $A(z)$ .

Noting the leading term of (26), in order to use the invertibility of  $S_2 + S_1v\mathcal{G}_1v^*S_1$  we invert  $-iB(z) + S_2$  on  $S_1L^2$ , and use Lemma 2.11 to invert  $-iB(z)$ , hence  $B(z)$ . Using the expansion (25) in (22) we have

$$\begin{aligned} -iB(z) + S_2 &= [S_2 + S_1v\mathcal{G}_1v^*S_1] + iz[S_1v\mathcal{G}_2v^*S_1 - S_1v\mathcal{G}_1v^*D_0v\mathcal{G}_1v^*S_1] + z^2\Gamma_6 \\ &\quad + z^3\Gamma_7 + z^4\Gamma_8 + \tilde{O}_2(z^{4+}). \end{aligned}$$

with  $\Gamma_i$  absolutely bounded operators independent of  $z$ .

We denote  $D_2 = (S_1v\mathcal{G}_1v^*S_1 + S_2)^{-1}$ . By Neumann series expansion for small  $|z|$  we have

$$\begin{aligned} (28) \quad (-iB(z) + S_2)^{-1} &= D_2 - izD_2[S_1v\mathcal{G}_2v^*S_1 - S_1v\mathcal{G}_1v^*D_0v\mathcal{G}_1v^*S_1]D_2 \\ &\quad + z^2\Gamma_9 + z^3\Gamma_{10} + z^4\Gamma_{11} + \tilde{O}_2(z^{4+}), \end{aligned}$$

where the  $\Gamma_i$ 's are absolutely bounded operators independent of  $z$ . Then, noting that  $S_1S_2 = S_2S_1 = S_2$ ,  $S_2D_2 = D_2S_2 = S_2$ ,

$$\begin{aligned} B_1(z) &:= \frac{S_2 - S_2(-iB(z) + S_2)^{-1}S_2}{z} \\ &= iS_2v\mathcal{G}_2v^*S_2 + S_2v\mathcal{G}_1v^*D_0v\mathcal{G}_1v^*S_2 + zS_2\Gamma_9S_2 + z^2S_2\Gamma_{10}S_2 + z^3S_2\Gamma_{11}S_2 + \tilde{O}_2(z^{3+}) \\ &= iS_2v\mathcal{G}_2v^*S_2 + zS_2\Gamma_9S_2 + z^2S_2\Gamma_{10}S_2 + z^3S_2\Gamma_{11}S_2 + \tilde{O}_2(z^{3+}). \end{aligned}$$

For the third equality we used that  $\mathcal{G}_1v^*S_2 = 0$ , (see Corollary 4.3). By Lemma 4.5, the operator  $S_2v\mathcal{G}_2v^*S_2$  is invertible on  $S_2L^2$ . Letting  $D_3 := (S_2v\mathcal{G}_2v^*S_2)^{-1}$  we have

$$(29) \quad B_1(z)^{-1} = -iD_3 + z\Gamma_{12} + z^2\Gamma_{13} + z^3\Gamma_{14} + \tilde{O}_2(z^{3+}).$$

Here  $\Gamma_i$ 's are finite rank operators since  $S_2$  is finite rank. Further, they are independent of  $z$ .

Using this expression in (23) for  $(-iB(z))^{-1} = iB^{-1}(z)$ , we have

$$B^{-1}(z) = -i(-iB(z) + S_2)^{-1} - \frac{i}{z} \left[ (-iB(z) + S_2)^{-1} S_2 (B_1(z))^{-1} S_2 (-iB(z) + S_2)^{-1} \right].$$

Plugging this in (23) we have,

$$(30) \quad A^{-1}(z) = (A(z) + S_1)^{-1} - \frac{i}{z} \left[ (A(z) + S_1)^{-1} S_1 (-iB(z) + S_2)^{-1} S_1 (A(z) + S_1)^{-1} \right] \\ - \frac{i}{z^2} \left[ (A(z) + S_1)^{-1} S_1 (-iB(z) + S_2)^{-1} S_2 B_1^{-1}(z) S_2 (-iB(z) + S_2)^{-1} S_1 (A(z) + S_1)^{-1} \right].$$

Inserting the expansions (25), (28), and (29) in this equality we obtain

$$A(z)^{-1} = -\frac{1}{z^2} S_2 D_3 S_2 + \frac{1}{z} \Omega + \Omega_0 + z\Omega_1 + \tilde{O}_2(z^{1+}) = -\frac{1}{z^2} S_2 D_3 S_2 + \frac{1}{z} \Omega + E(z).$$

Here  $\Omega_j$ 's are absolutely bounded operators independent of  $z$ . Also,  $\Omega$  is a finite rank operator. Note that by (30),  $\Omega$  is the sum of a composition of  $z$  independent operators, at least one of which is  $S_1$  or  $S_2$ . The fact that  $S_1$  and  $S_2$  are finite rank operators establishes the claim.  $\square$

### 3. DISPERSIVE ESTIMATES

In this section we prove Theorems 1.1 and 1.2 through a careful analysis of the oscillatory integrals that naturally arise in the Stone's formula (7). We divide this into three subsections. First, in Subsection 3.1, we consider the Born series terms and show that they satisfy the bound  $\langle t \rangle^{-\frac{3}{2}}$  as an operator from  $L^1(\mathbb{R}^3) \rightarrow L^\infty(\mathbb{R}^3)$ . In Subsections 3.2 and 3.3, we show that the singular terms that arise in the expansion of the spectral measure when there are threshold resonances or eigenvalues yield a slower time decay rate, but are finite rank operators.

Recall the expansion (19) for the perturbed resolvent. To emphasize the change of variables and dependence now on the spectral parameter  $z$ , we write the resolvents as  $\mathcal{R}_0(z)$  rather than  $\mathcal{R}_0(\lambda)$ . Under this identification, we have  $\mathcal{R}_0^-(z) = \mathcal{R}_0^+(-z)$ . Without loss of generality, we take  $t > 0$ , the proof for  $t < 0$  requires only minor adjustments. We consider integrals of the form below for the contribution of the finite terms of the Born series (19) to the Stone's formula (7).

$$\int_m^\infty e^{-it\lambda} \chi(\lambda) \left[ \mathcal{R}_0^+(z) (V\mathcal{R}_0^+(z))^k - \mathcal{R}_0^-(z) (V\mathcal{R}_0^-(z))^k \right] d\lambda.$$

Recall that  $\lambda = \sqrt{z^2 + m^2}$ , we can re-write this as

$$(31) \quad \int_0^\infty e^{-it\sqrt{z^2 + m^2}} \frac{z\chi(z)}{\sqrt{z^2 + m^2}} \left[ \mathcal{R}_0^+(z) (V\mathcal{R}_0^+(z))^k - \mathcal{R}_0^-(z) (V\mathcal{R}_0^-(z))^k \right] dz.$$

We utilize from the following consequence of the classical Van der Corput lemma, [36].

**Lemma 3.1.** *If  $\phi : [a, b] \rightarrow \mathbb{R}$  obeys the bound  $|\phi''(z)| \geq t > 0$  for all  $z \in [a, b]$ , and if  $\psi : [a, b] \rightarrow \mathbb{C}$  such that  $\psi' \in L^1([a, b])$ , then*

$$\left| \int_a^b e^{i\phi(z)} \psi(z) dz \right| \lesssim t^{-\frac{1}{2}} \left\{ |\psi(b)| + \int_a^b |\psi'(z)| dz \right\}.$$

**3.1. The Born Series.** We have the following lemma for the finite terms of Born series.

**Proposition 3.2.** *Let  $|V(x)| \lesssim \langle x \rangle^{-3-}$ . Then for any  $k \in \mathbb{N} \cup \{0\}$ , the following bound holds*

$$(32) \quad \sup_{x, y \in \mathbb{R}^3} \left| \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} \left[ \mathcal{R}_0^+(V\mathcal{R}_0^+)^k - \mathcal{R}_0^-(V\mathcal{R}_0^-)^k \right](z)(x, y) dz \right| \lesssim \langle t \rangle^{-\frac{3}{2}}.$$

We use the algebraic identity

$$(33) \quad \mathcal{R}_0^+(V\mathcal{R}_0^+)^k - \mathcal{R}_0^-(V\mathcal{R}_0^-)^k = \sum_{\ell=0}^k (\mathcal{R}_0^-V)^\ell [\mathcal{R}_0^+ - \mathcal{R}_0^-] (V\mathcal{R}_0^+)^{k-\ell}.$$

**Lemma 3.3.** *We have the following bounds on the first derivative of the difference of free resolvents.*

$$[\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x, y) = \tilde{O}_1(z).$$

Furthermore,

$$(34) \quad \begin{aligned} \partial_z [\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x - y) &= \frac{i}{2\pi} \left( \frac{\alpha \cdot (x - y)}{|x - y|} \right) \sin(z|x - y|) \\ &+ \frac{z}{2\pi\sqrt{z^2+m^2}} \frac{\sin(z|x - y|)}{|x - y|} + (m\beta + \sqrt{z^2+m^2}I) \frac{\cos(z|x - y|)}{2\pi}. \end{aligned}$$

*Proof.* Note that

$$(35) \quad \begin{aligned} [\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x, y) &= \frac{1}{4\pi} [-i\alpha \cdot \nabla + m\beta + \sqrt{m^2+z^2}] \left[ \frac{e^{iz|x-y|} - e^{-iz|x-y|}}{|x-y|} \right] \\ &= \frac{1}{2\pi} \alpha \cdot \nabla \left[ \frac{\sin(z|x-y|)}{|x-y|} \right] + \frac{i}{2\pi} (m\beta + \sqrt{z^2+m^2}I) \left[ \frac{\sin(z|x-y|)}{|x-y|} \right]. \end{aligned}$$

Using this representation, we express the difference of free resolvents with two pieces. We ignore the constant factors. We first consider  $A(z, |x-y|) := \alpha \cdot \nabla \left[ \frac{\sin(z|x-y|)}{|x-y|} \right]$ , which satisfies the bound  $\tilde{O}_1(z^2)$ . By direct computation, we have

$$A(z, |x-y|) = \left[ \alpha \cdot \frac{(x-y)}{|x-y|} \right] \left[ \frac{z|x-y| \cos(z|x-y|) - \sin(z|x-y|)}{|x-y|^2} \right].$$

First if  $z|x-y| \gtrsim 1$ , using  $|x-y|^{-1} \lesssim z$  establishes the desired bound. To see the inequality for  $z|x-y| \lesssim 1$  note that by Taylor series expansion one has  $s \cos(s) - \sin(s) = \tilde{O}_1(s^3)$ . Taking derivative of  $A(z, |x-y|)$  we have

$$\partial_z A(z, |x-y|) = \left( \frac{\alpha \cdot (x-y)}{|x-y|} \right) z \sin(z|x-y|).$$

The desired bound easily follows from this explicit representation.

We move to the second part of (35) let  $B(z, |x - y|) := (m\beta + \sqrt{z^2 + m^2}I) \frac{\sin(z|x - y|)}{|x - y|}$ . A direct computation shows

$$\partial_z B(z, |x - y|) = \frac{z}{\sqrt{z^2 + m^2}} \frac{\sin(z|x - y|)}{|x - y|} + (m\beta + \sqrt{z^2 + m^2}I) \cos(z|x - y|).$$

As before, considering the cases of  $z|x - y| \gtrsim 1$  and  $z|x - y| \lesssim 1$  separately suffices.  $\square$

*Proof of Proposition 3.2.* Using the identity (33), we fix  $\ell$  and consider the contribution of

$$(36) \quad \left| \int_0^\infty e^{-it\sqrt{z^2 + m^2}} \frac{z\chi(z)}{\sqrt{z^2 + m^2}} \left[ (\mathcal{R}_0^- V)^\ell [\mathcal{R}_0^+ - \mathcal{R}_0^-] (V\mathcal{R}_0^+)^{k-\ell} \right] (z)(x_0, x_k) dz \right|.$$

For notational convenience let  $J = \{0, 1, 2, \dots, k\} \setminus \{\ell\}$ ,  $J^- = \{0, 1, \dots, \ell - 1\}$  and  $J^+ = \{\ell + 1, \ell + 2, \dots, k\}$ . Note that one of  $J^-$  or  $J^+$  may be empty. We first establish that integral is bounded. Using the expansion (10), we have (when  $0 < z \ll 1$ )

$$(37) \quad \begin{aligned} \mathcal{R}_0^\pm(z)(x, y) &= [-i\alpha \cdot \nabla + m\beta + \sqrt{m^2 + z^2}I] \frac{e^{\pm iz|x - y|}}{4\pi|x - y|} \\ &= \left[ \left( \frac{\alpha \cdot (x - y)}{|x - y|} \right) \left[ \pm iz + \frac{1}{|x - y|} \right] + \left( m\beta + \sqrt{z^2 + m^2}I \right) \right] \frac{e^{\pm iz|x - y|}}{4\pi|x - y|} \\ &= e^{\pm iz|x - y|} H_1(z, x, y), \quad \sup_{|z| < z_0} |\partial_z^k H_1(z, x, y)| \lesssim \frac{1}{|x - y|} + \frac{1}{|x - y|^2}, \end{aligned}$$

for each  $k = 0, 1, 2, \dots$ . Furthermore,

$$(38) \quad \begin{aligned} \partial_z \mathcal{R}_0^\pm(z)(x, y) &= \left[ iz \frac{\alpha \cdot (x - y)}{|x - y|} \pm im\beta \pm i\sqrt{z^2 + m^2}I + \frac{z}{|x - y|\sqrt{z^2 + m^2}} \right] \frac{e^{iz|x - y|}}{4\pi} \\ &= e^{\pm iz|x - y|} H_2(z, x, y), \quad \sup_{|z| < z_0} |\partial_z^k H_2(z, x, y)| \lesssim 1 + \frac{1}{|x - y|} \quad k = 0, 1, 2, \dots \end{aligned}$$

From this we see, for  $0 < z \ll 1$ ,

$$(39) \quad |\partial_z^j \mathcal{R}_0^\pm(z)(x, y)| \lesssim \left( \frac{1}{|x - y|} + \frac{1}{|x - y|^2} \right) |x - y|^j, \quad j = 0, 1, 2.$$

Using this bound and (36), the  $z$  integral is clearly bounded due to the cut-off to  $0 < z \ll 1$ ,

$$\begin{aligned} &\sup_{x_0, x_k \in \mathbb{R}^3} \left| \int_0^\infty e^{-it\sqrt{z^2 + m^2}} \frac{z\chi(z)}{\sqrt{z^2 + m^2}} \left[ (\mathcal{R}_0^- V)^\ell [\mathcal{R}_0^+ - \mathcal{R}_0^-] (V\mathcal{R}_0^+)^{k-\ell} \right] (x_0, x_k) dz \right| \\ &\lesssim \sup_{x_0, x_k \in \mathbb{R}^3} \int_{\mathbb{R}^{3k}} \prod_{p=1}^k |V(x_p)| \prod_{j \in J} \left( \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) dx_1 dx_2 \dots dx_k. \end{aligned}$$

This is seen to be bounded uniformly in  $x_0, x_k$  using Lemma 2.3 to iterate the bound

$$\sup_{x_{j+1} \in \mathbb{R}^3} \int_{\mathbb{R}^3} \langle x_j \rangle^{-2-} \left( \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) dx_j \lesssim 1,$$

first integrating in  $x_\ell$ .



To establish the time decay, we integrate by parts once then use Lemma 3.1. Integrating by parts once leaves us to bound

$$\frac{1}{t} \int_0^\infty e^{-it\sqrt{z^2+m^2}} \partial_z \left( \chi(z) (\mathcal{R}_0^- V)^\ell [\mathcal{R}_0^+ - \mathcal{R}_0^-] (V \mathcal{R}_0^+)^{k-\ell} \right) (z) dz.$$

Note that there is no boundary term since  $[\mathcal{R}_0^+ - \mathcal{R}_0^-] = \tilde{O}_1(z)$  by Lemma 3.3 and by Lemma 2.1 the free resolvents are bounded with respect to  $z$ , and the support of  $\chi$ . We consider two cases, if the derivative acts on the difference of resolvents or on a resolvent. If the derivative acts on the cut-off function, we can easily integrate by parts again with the existing bounds. We first consider when the derivative acts on difference of resolvents. From the representation in (34), we can write

$$\begin{aligned} \partial_z [\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x_\ell, x_{\ell+1}) &= e^{iz|x_\ell - x_{\ell+1}|} A_1(z, |x_\ell - x_{\ell+1}|) \\ &\quad + e^{-iz|x_\ell - x_{\ell+1}|} A_2(z, |x_\ell - x_{\ell+1}|) + \tilde{O}_1(z), \end{aligned}$$

with

$$|\partial_z^j A_1(z, |x_\ell - x_{\ell+1}|)|, \quad |\partial_z^j A_2(z, |x_\ell - x_{\ell+1}|)| \lesssim 1, \quad j = 0, 1.$$

The error term comes because we have  $\left[ \frac{z}{\sqrt{z^2+m^2}} \frac{\sin(zr)}{r} \right] = \tilde{O}_1(z)$ . With a slight abuse of notation, we denote both the operators  $A_1$  and  $A_2$  by  $a(z)$ . Combining this with (37), we need to bound terms of the form

$$\begin{aligned} \frac{1}{t} \int_0^\infty e^{-it\sqrt{z^2+m^2}} \chi(z) \left( e^{\pm iz|x_\ell - x_{\ell+1}|} a(z) + \tilde{O}_1(z) \right) \\ \prod_{j \in J^-} e^{-iz|x_j - x_{j+1}|} H_1(z, x_j, x_{j+1}) \prod_{p \in J^+} e^{iz|x_p - x_{p+1}|} H_1(z, x_p, x_{p+1}) dz. \end{aligned}$$

We apply Lemma 3.1 with

$$\phi(z) = -t\sqrt{z^2+m^2} - z \left( \sum_{j \in J^-} |x_j - x_{j+1}| + \gamma |x_\ell - x_{\ell+1}| - \sum_{p \in J^+} |x_p - x_{p+1}| \right),$$

where  $\gamma \in \{-1, 0, 1\}$ , and

$$\psi(z) = [a(z) + \tilde{O}_1(z)] \prod_{j \in J} H_1(z, x_j, x_{j+1}).$$

We may again bound the contribution of the spatial integrals by Lemma 2.3.

$$\frac{1}{t^{\frac{3}{2}}} \int_{\mathbb{R}^{3k}} \prod_{p=1}^k |V(x_p)| \prod_{j \in J} \left( \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) dx_1 dx_2 \dots dx_k \lesssim \frac{1}{t^{\frac{3}{2}}}.$$

On the other hand, if the derivative hits one of the iterated resolvents, we have to bound

$$\frac{1}{t} \int_0^\infty e^{-it\sqrt{z^2+m^2}} (\mathcal{R}_0^- V)^\ell [\mathcal{R}_0^+ - \mathcal{R}_0^-] \partial_z (V \mathcal{R}_0^+)^{k-\ell} (z) dz.$$

Using Lemma 3.3, we have  $[\mathcal{R}_0^+ - \mathcal{R}_0^-](z)(x_\ell, x_{\ell+1}) = \tilde{O}_1(z)$ . Then, using (37), we have

$$(\mathcal{R}_0^- V)^\ell \partial_z (V \mathcal{R}_0^+)^{k-\ell}(z) = e^{iz(\sum_{p \in J^+} |x_p - x_{p+1}| - \sum_{j \in J^-} |x_j - x_{j+1}|)} b(z),$$

where

$$|b(z)|, |\partial_z b(z)| \lesssim \sum_{\ell \in J^+} |x_\ell - x_{\ell+1}| \prod_{j \in J} \left( \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) \prod_{r=1}^k V(x_r).$$

Combining these bounds we have to bound

$$\frac{1}{t} \int_0^\infty e^{-it\sqrt{z^2+m^2}+iz(\sum_{p \in J^+} |x_p - x_{p+1}| - \sum_{j \in J^-} |x_j - x_{j+1}|)} \psi(z) dz,$$

where  $\psi(z), \psi'(z)$  are supported on a small neighborhood of  $z = 0$  and satisfy

$$|\psi(z)|, |\partial_z \psi(z)| \lesssim \sum_{\ell \in J^+} |x_\ell - x_{\ell+1}| \prod_{j \in J} \left( \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) \prod_{r=1}^k V(x_r).$$

Thus, we apply Lemma 3.1 to bound the spatial integral

$$\sup_{x_0, x_k \in \mathbb{R}^3} \frac{1}{t^{\frac{3}{2}}} \int_{\mathbb{R}^{3k}} \sum_{\ell \in J^+} |x_\ell - x_{\ell+1}| \prod_{j \in J} \left( \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) \prod_{r=1}^k \langle x_r \rangle^{-3-} dx_1 dx_2 \dots dx_k.$$

Using Lemma 2.3, first in  $x_\ell$ , we show that the spatial integrals are bounded uniformly in  $x_0, x_{k+1}$  by iterating the bound

$$\sup_{x_{j+1} \in \mathbb{R}^3} \int_{\mathbb{R}^3} \langle x_j \rangle^{-3-} \left( 1 + \frac{1}{|x_j - x_{j+1}|} + \frac{1}{|x_j - x_{j+1}|^2} \right) dx_j \lesssim 1.$$

□

We finish this subsection with the following general lemma which will be useful in the following subsections.

**Lemma 3.4.** *Assume that the operator  $E(z)$  with kernel  $E(z)(x, y)$  satisfies (for  $0 < |z| < z_0$ )*

$$|||\partial_z^k E(z)(x, y)|||_{L^2 \rightarrow L^2} \lesssim 1, \quad k = 0, 1, \quad \text{and} \quad |||\partial_z^2 E(z)(x, y)|||_{L^2 \rightarrow L^2} \lesssim z^{-1+}.$$

*Also assume that the operators  $E_1(z)$  and  $E_2(z)$  satisfy (for some  $\alpha \geq 0$ )*

$$|\partial_z^k E_j(z)(x, y)| \lesssim (|x - y|^{-2} + |x - y|^\alpha), \quad j = 0, 1, \quad k = 0, 1, \quad \text{and}$$

$$|\partial_z^2 E_j(z)(x, y)| \lesssim z^{-1+} (|x - y|^{-2} + |x - y|^\alpha), \quad j = 0, 1.$$

*Let  $|V(x)| \lesssim \langle x \rangle^{-\beta}$  for some  $\beta > 2\alpha + 3$ . Then,*

$$\sup_{x, y \in \mathbb{R}^3} \left| \int_{-\infty}^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{m^2+z^2}} \left( \mathcal{R}_0 V E_1 v^* E v E_2 V \mathcal{R}_0 \right) (z)(x, y) dz \right| \lesssim \langle t \rangle^{-\frac{3}{2}}.$$

*Proof.* We start with bound for small  $t$ . Using the bounds in the hypothesis for  $k = 0$  and using  $|\mathcal{R}_0(z)(x, y)| \lesssim 1 + |x - y|^{-2}$  from (37), we estimate the  $z$  integral by

$$\int_{-\infty}^{\infty} \chi(z) \psi(z) dz, \quad \text{where}$$

$$\psi(z) = \int_{\mathbb{R}^{12}} \frac{(1 + r_1^{-2})(r_2^{-2} + r_2^\alpha)}{\langle x_1 \rangle^\beta \langle x_2 \rangle^{\frac{\beta}{2}}} |E(z)(x_2, y_2)| \frac{(r_3^{-2} + r_3^\alpha)(1 + r_4^{-2})}{\langle y_2 \rangle^{\frac{\beta}{2}} \langle y_1 \rangle^\beta} dx_1 dx_2 dy_1 dy_2.$$

Here  $r_1 := |x - x_1|, r_2 := |x_1 - x_2|, r_3 := |y_2 - y_1|, r_4 := |y_1 - y|$ . We can bound  $\psi$  by

$$\left\| \int_{\mathbb{R}^3} \frac{(1 + r_1^{-2})(r_2^{-2} + r_2^\alpha)}{\langle x_1 \rangle^\beta \langle x_2 \rangle^{\frac{\beta}{2}}} dx_1 \right\|_{L_{x_2}^2(\mathbb{R}^3)}^2 \| |E(z)| \|_{L^2 \rightarrow L^2} \left\| \int_{\mathbb{R}^3} \frac{(1 + r_4^{-2})(r_3^{-2} + r_3^\alpha)}{\langle y_1 \rangle^\beta \langle y_2 \rangle^{\frac{\beta}{2}}} dy_1 \right\|_{L_{y_2}^2(\mathbb{R}^3)}^2.$$

Note that using Lemma 2.3

$$\begin{aligned} \int_{\mathbb{R}^3} \frac{(1 + r_1^{-2})(r_2^{-2} + r_2^\alpha)}{\langle x_1 \rangle^\beta \langle x_2 \rangle^{\frac{\beta}{2}}} dx_1 &\lesssim \int_{\mathbb{R}^3} \frac{1}{\langle x_1 \rangle^{\beta-\alpha} \langle x_2 \rangle^{\frac{\beta}{2}-\alpha}} dx_1 + \int_{\mathbb{R}^3} r_1^{-2} r_2^{-2} \langle x_2 \rangle^{-\frac{\beta}{2}} dx_1 \\ &\lesssim \langle x_2 \rangle^{-\frac{\beta}{2}+\alpha} + |x - x_2|^{-1} \langle x_2 \rangle^{-\frac{\beta}{2}} \in L_{x_2}^2, \end{aligned}$$

uniformly in  $x$  provided that  $\beta > 2\alpha + 3$ . This finishes the proof since  $\| |E(z)| \|_{L^2 \rightarrow L^2}$  is bounded on the support of  $\chi$ .

Now we consider the claim for large  $t$ . After an integration by parts we have to bound

$$\frac{1}{t} \int_{-\infty}^{\infty} e^{it\sqrt{z^2+m^2}} \partial_z \left( \chi(z) [\mathcal{R}_0 V E_1 v^* E v E_2 V \mathcal{R}_0](x, y) \right) dz.$$

Now using Lemma 3.1 with the phase  $\phi = t\sqrt{z^2+m^2} + zr_1 + zr_4$  we estimate the integral above by

$$\frac{1}{|t|^{\frac{3}{2}}} \int_{-\infty}^{\infty} \left| \partial_z \left[ e^{-iz(r_1+r_4)} \partial_z \left( \chi(z) [\mathcal{R}_0 V E_1 v^* E v E_2 V \mathcal{R}_0](x, y) \right) \right] \right| dz.$$

Note that using (37) and (38) we have

$$|\mathcal{R}_0|, |\partial_z \mathcal{R}_0|, |\partial_z e^{-iz|x-y|} \mathcal{R}_0|, |\partial_z e^{-iz|x-y|} \partial_z \mathcal{R}_0| \lesssim 1 + |x - y|^{-2}.$$

The proof now follows from the calculation above for small  $t$ ; the only difference is, if both derivatives hit  $E$  (or one of  $E_1, E_2$ ), the  $z$  integral will have a harmless  $z^{-1+}$  term, which is integrable on the support of  $\chi(z)$ .  $\square$

**3.2. Dispersive estimates when there is a resonance of the first kind.** In this subsection we consider the case when there is a resonance of the first kind at threshold energy, that is when  $S_1 \neq 0$  and  $S_2 = 0$ , in which case  $S_1$  is rank at most two by Corollary 4.4.

In the previous section we established the contribution of the first three terms in the expansion (19) to the Stone's formula (7). Now we turn to the last term in (19), we need to analyze

$$(40) \quad \int_0^\infty e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} \left[ [\mathcal{R}_0^+ V \mathcal{R}_0^+ v^* (A^+)^{-1} v \mathcal{R}_0^+ V \mathcal{R}_0^+](z) \right]$$

$$- [\mathcal{R}_0^- V \mathcal{R}_0^- v^* (A^-)^{-1} v \mathcal{R}_0^- V \mathcal{R}_0^-](z) dz.$$

Recalling the discussion immediately preceding Lemma 2.7, we identify  $\mathcal{R}_0^-(-z) = \mathcal{R}_0^+(z) =: \mathcal{R}_0(z)$ . Similarly,  $A^-(-z) = A^+(z) := A(z)$ . Hence, by a change of variable we can extend the integral (40) to the whole real line and obtain

$$(40) = \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0 V \mathcal{R}_0 v^* A^{-1} v \mathcal{R}_0 V \mathcal{R}_0](z)(x, y) dz.$$

In contrast to the analysis of the Born series in the previous subsection, we extend the integral to the real line. This will allow us to integrate by parts without boundary terms and, after a change of variables, use Fourier transform techniques.

Note that we have

$$(41) \quad \sup_{x, y \in \mathbb{R}^3} \left| \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0 V \mathcal{R}_0 v^* A^{-1} v \mathcal{R}_0 V \mathcal{R}_0](z)(x, y) dz \right| \\ \lesssim \sup_{x, y, |z| \leq z_0} |[\mathcal{R}_0(z) V \mathcal{R}_0(z) v^* [z A^{-1}(z)] v \mathcal{R}_0(z) V \mathcal{R}_0(z)](x, y)|.$$

By Lemma 2.13,  $|z| \|A^{-1}(z)(x, y)\|_{L^2 \rightarrow L^2} \lesssim 1$  on the support of  $\chi$ . Then, by Remark 2.4 we have

$$|(41)| \lesssim \sup_{x, y \in \mathbb{R}^3} \|[\mathcal{R}_0 V \mathcal{R}_0 v^*](x, x_2)\|_{L_{x_2}^2} \|z A^{-1}(z)\|_{L^2 \rightarrow L^2} \|v \mathcal{R}_0 V \mathcal{R}_0\|_{L_{y_2}^2} \lesssim 1,$$

which shows the boundedness of (41) as  $t \rightarrow 0$ . Hence, to establish the claim of Theorem 1.1, it will be enough to prove the following proposition for any  $t > 1$ .

**Proposition 3.5.** *Under the assumptions of Theorem 1.1, we have*

$$\int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0 V \mathcal{R}_0 v^* A^{-1} v \mathcal{R}_0 V \mathcal{R}_0](z)(x, y) dz = t^{-\frac{1}{2}} e^{-imt} K_t(x, y) + O(t^{-\frac{3}{2}}),$$

where the error term holds uniformly in  $x, y$ ;  $K_t(x, y) = P_r(x, y) + \tilde{K}_t(x, y)$  is a time dependent operator of rank at most 2 satisfying  $\sup_t \|K_t\|_{L^1 \rightarrow L^\infty} \lesssim 1$  and  $|\tilde{K}_t(x, y)| \lesssim \langle x \rangle^j \langle y \rangle^j \langle t \rangle^{-j}$  for any  $0 \leq j \leq 1$ . Moreover,

$$P_r(x, y) = \sum_{j=1}^2 c_j \psi_j(x) \psi_j^*(y), \text{ where } c_j = \frac{(-2\pi i)^{\frac{3}{2}}}{m^{\frac{3}{2}} \|M_{uc} V \psi_j\|_{\mathbb{C}^4}^2} \text{ and} \\ \psi_j \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3) \cap L^\infty(\mathbb{R}^3), \quad (D_m + V - mI)\psi_j = 0, \\ \langle M_{uc} V \psi_i, M_{uc} V \psi_j \rangle = \|M_{uc} V \psi_j\|_{\mathbb{C}^4}^2 \delta_{ij}, \quad i, j = 1, 2.$$

Here  $c_2 = 0$  iff  $\text{rank}(S_1) = 1$ .

To establish Proposition 3.5, using the expansion in Lemma 2.13, it suffices to consider the following integrals

$$(42) \quad \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{-i\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0(z)V\mathcal{R}_0(z)v^*S_1D_1S_1v\mathcal{R}_0(z)V\mathcal{R}_0(z)](x,y)dz, \\ \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0(z)V\mathcal{R}_0(z)v^*E(z)v\mathcal{R}_0(z)V\mathcal{R}_0(z)](x,y)dz.$$

The second integral is  $O(\langle t \rangle^{-3/2})$  using Lemma 3.4 provided that  $\beta > 5$ . Indeed, the required bound for  $E$  is given in Lemma 2.13, and for  $E_1 = E_2 = \mathcal{R}_0$  the hypothesis is satisfied with  $\alpha = 1$  using (39).

Now we consider the first integral in (42). Using (10) for  $\mathcal{R}_0(z)$  and (8), and letting  $F(x, y) := \frac{1}{4\pi}[i\alpha \cdot \frac{(x-y)}{|x-y|^2} + 2mI_{uc}]$ , we have

$$(43) \quad \mathcal{R}_0(z)(x, y) = F(x, y) \frac{e^{iz|x-y|}}{|x-y|} + \left[ iz\alpha \cdot \frac{(x-y)}{|x-y|} + (\sqrt{z^2+m^2} - m)I \right] \frac{e^{iz|x-y|}}{4\pi|x-y|}.$$

Hence,

$$(44) \quad \mathcal{R}_0(z)(x, x_1)\mathcal{R}_0(z)(x_1, x_2)\mathcal{R}_0(z)(y_2, y_1)\mathcal{R}_0(z)(y_1, y) \\ = F(x, x_1)F(x_1, x_2)F(y_2, y_1)F(y_1, y) \frac{e^{iz\theta}}{r_1r_2r_3r_4} + z\mathcal{E}(z)e^{iz\theta},$$

where  $\theta = |x - x_1| + |x_1 - x_2| + |y_2 - y_1| + |y_1 - y| := r_1 + r_2 + r_3 + r_4$  and  $\mathcal{E}(z)$  satisfies the bound

$$|\mathcal{E}^{(j)}(z)| \lesssim \prod_{i=1}^4 \left( \frac{1}{r_i^2} + \frac{1}{r_i} \right), \quad j = 0, 1, 2.$$

Therefore, for the first term in (42) is given by

$$(45) \quad \frac{F(x, x_1)F(x_1, x_2)F(y_2, y_1)F(y_1, y)}{r_1r_2r_3r_4} \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}+iz\theta} \frac{\chi(z)}{\sqrt{z^2+m^2}} dz \\ + \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} \mathcal{E}(z)e^{iz\theta} dz = I + II.$$

Note that  $II$  can be estimated as follows using integration by parts followed with Lemma 3.1,

$$|II| = \left| \frac{C}{t} \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}+iz\theta} \left[ (\chi(z)\mathcal{E}(z))' + \theta\chi(z)\mathcal{E}(z) \right] dz \right| \\ \lesssim \frac{1}{|t|^{3/2}} \left| \int_{-1}^1 (\chi(z)\mathcal{E}(z))'' + \theta(\chi(z)\mathcal{E}(z))' dz \right| \lesssim \frac{1}{|t|^{3/2}} \langle \max_i r_i \rangle \prod_{i=1}^4 \left( \frac{1}{r_i^2} + \frac{1}{r_i} \right).$$

The spatial integrals can be estimated as in the proof of Lemma 3.4 with  $\alpha = 0$ ,  $\beta > 3$ , and  $E = S_1D_1S_1$ .

Next we consider the first term in (45). Note that this integral can be estimated by  $t^{-\frac{1}{2}}$  easily using Lemma 3.1 with  $\phi(z) = -t\sqrt{z^2 + m^2} + z\theta$ . In the rest of this subsection we establish the properties of the operator which has decay rate  $t^{-\frac{1}{2}}$ .

For notational convenience, we suppress the integral kernels' spatial variable dependence, which should be clear from context. First we assume that at least one of the  $r_j$ 's is greater than  $t$ . In this case we have  $\frac{1}{\max_j r_j} \lesssim \frac{1}{t}$ . Hence, we can exchange the largest  $r_j$  with  $t$  to gain extra time decay. Using an analysis similar to that in the proof of Lemma 3.4 one can easily see that the spatial integrals converge. Thus, we have

$$|I| \lesssim \frac{1}{t^{\frac{1}{2}}} \left| \int_{\mathbb{R}^{12}} \frac{FVFv^*S_1D_1S_1vFVF}{r_1r_2r_3r_4} dx_1dx_2dy_1dy_2 \right| \lesssim t^{-\frac{3}{2}}.$$

Now it remains to consider the case when  $r_j \ll t$  for all  $j$ . We start with the following lemma.

**Lemma 3.6.** *Let  $\theta = \sum_{j=1}^4 r_j$ . Then,*

$$(46) \quad \prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right) \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}+iz\theta} \frac{\chi(z)}{\sqrt{z^2+m^2}} dz = \frac{(-2\pi i)^{\frac{1}{2}} e^{3imt}}{(mt)^{\frac{1}{2}}} \prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right) e^{-im(t^2-r_j^2)^{\frac{1}{2}}} \\ + O\left(\frac{1}{t^{\frac{3}{2}}} \left[ \sum_{1 \leq i < j \leq 4} r_i r_j + \sum_{j=1}^4 r_j + 1 \right]\right).$$

For the proof of Lemma 3.6 we need the following lemma.

**Lemma 3.7.** *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be  $C^2$  with bounded derivatives. Then for any  $a_i > 0$  we have*

$$f\left(\sum_{j=1}^n a_j\right) = \sum_{j=1}^n f(a_j) - (n-1)f(0) + O\left(\sum_{1 \leq i < j \leq n} a_i a_j\right).$$

*Proof.* By a simple induction argument, it suffices to prove this for  $n = 2$ . Without loss of generality we can also assume that  $a_2 \geq a_1$ . By the mean value theorem, we have

$$\begin{aligned} f(a_1 + a_2) &= f(a_1) + f(a_2) + [f(a_1 + a_2) - f(a_2)] - [f(a_1) - f(0)] - f(0) \\ &= f(a_1) + f(a_2) + f'(c_1)a_1 - f'(c_2)a_1 - f(0) \quad \text{for } c_1 \in (a_2, a_2 + a_1), c_2 \in (0, a_1) \\ &= f(a_1) + f(a_2) + a_1(c_1 - c_2)f''(c) - f(0). \end{aligned}$$

Since  $0 \leq a_2 - a_1 \leq c_1 - c_2 \leq a_1 + a_2 \leq 2a_2$ , this yields the lemma.  $\square$

*Proof of Lemma 3.6.* For the sake of simplicity we prove the lemma for  $m = 1$ . Note that first, the critical point of  $\phi(z) = \sqrt{z^2 + 1} - z\gamma$  is  $\omega = \frac{\gamma}{\sqrt{1-\gamma^2}}$ . Here  $\omega$  is defined since  $r_j \ll t$  for all  $j$  implies  $\gamma = \frac{\theta}{t} \ll 1$ . We use the change of variables  $z \mapsto z + \omega$  to move the critical point to zero and write

$$(46) = \prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right) e^{-it\sqrt{1-\gamma^2}} \int_{-\infty}^{\infty} e^{-it\left(\sqrt{(z+\omega)^2+1}-z\gamma-\frac{1}{\sqrt{1-\gamma^2}}\right)} \frac{\chi(z+\omega)}{\sqrt{(z+\omega)^2+1}} dz.$$

With a change of variable  $z = g(s) = \frac{s}{\sqrt{1-\gamma^2}}(\sqrt{1 + \frac{s^2}{4}} + \frac{s\gamma}{2})$  this integral can be written as

$$(46) = \prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right) e^{-it\sqrt{1-\gamma^2}} \int_{-\infty}^{\infty} e^{-its^2 \frac{\sqrt{1-\gamma^2}}{2}} \psi(s) ds,$$

where

$$\psi(s) := \frac{\chi(g(s) + \omega)}{\sqrt{(g(s) + \omega)^2 + 1}} g'(s).$$

Note that  $\psi$  is supported on  $\{s : |s| \lesssim 1\}$ . Since on this set  $|g^{(k)}(s)| \lesssim 1$  for all  $k \geq 0$ , we see that  $\psi$  is a Schwartz function with derivatives bounded uniformly in  $\gamma \ll 1$ . Then, we have

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-its^2 \frac{\sqrt{1-\gamma^2}}{2}} \psi(s) ds &= (-2\pi i)^{\frac{1}{2}} \int_{-\infty}^{\infty} \mathcal{F}^{-1}\left(\frac{e^{-it\frac{\sqrt{1-\gamma^2}}{2}(\cdot)^2}}{(1-\gamma^2)^{\frac{1}{4}}t^{\frac{1}{2}}}\right)(\xi) \widehat{\psi}(\xi) d\xi \\ &= \frac{(-2\pi i)^{\frac{1}{2}}}{(1-\gamma^2)^{\frac{1}{4}}t^{\frac{1}{2}}} \int_{-\infty}^{\infty} e^{-i\frac{2\xi^2}{t\sqrt{1-\gamma^2}}} \widehat{\psi}(\xi) d\xi \\ &= \frac{(-2\pi i)^{\frac{1}{2}}}{(1-\gamma^2)^{\frac{1}{4}}t^{\frac{1}{2}}} \left[ \psi(0) + \int_{-\infty}^{\infty} \left[ e^{-i\frac{2\xi^2}{t\sqrt{1-\gamma^2}}} - 1 \right] \widehat{\psi}(\xi) d\xi \right]. \end{aligned}$$

Note that for  $\gamma \ll 1$  we have  $(1-\gamma^2)^{-\frac{1}{4}} \lesssim 1$ , and

$$\begin{aligned} \frac{1}{(1-\gamma^2)^{\frac{1}{4}}t^{\frac{1}{2}}} \left| \int_{-\infty}^{\infty} \left[ e^{-i\frac{2\xi^2}{t\sqrt{1-\gamma^2}}} - 1 \right] \widehat{\psi}(\xi) d\xi \right| &\lesssim \frac{1}{t^{\frac{3}{2}}} \int_{-\infty}^{\infty} |\xi^2 \widehat{\psi}(\xi)| d\xi \\ &\lesssim \frac{1}{t^{\frac{3}{2}}} \|\psi\|_{L^1} + \frac{1}{t^{\frac{3}{2}}} \|\psi''''\|_{L^1} \lesssim \frac{1}{t^{\frac{3}{2}}}. \end{aligned}$$

Hence, this term has the contribution  $O(t^{-\frac{3}{2}})$  to (46). For the last equality we used the fact that  $\|\partial_z^k \psi\|_{L^1} \lesssim 1$  uniformly in  $\gamma$ .

We are left with the contribution of  $\psi(0)$  to (46) which is given by

$$\prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right) t^{-\frac{1}{2}} e^{-it\sqrt{1-\gamma^2}} \frac{\chi(\omega)}{\sqrt{\omega^2 + 1}(1-\gamma^2)^{\frac{3}{4}}} = t^{-\frac{1}{2}} \prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right) e^{-itf(\gamma)} \frac{\chi\left(\frac{\gamma}{\sqrt{1-\gamma^2}}\right)}{(1-\gamma^2)^{\frac{1}{4}}},$$

where  $f(\gamma) = \sqrt{1-\gamma^2}\chi(\gamma/4)$  with  $\gamma = \frac{\theta}{t}$ . Note that  $f$  has bounded derivatives. Since  $f(0) = 1$ , using Lemma 3.7 we obtain (in the support of  $\prod_{j=1}^4 \chi\left(\frac{r_j}{t}\right)$ )

$$\begin{aligned} (47) \quad e^{-itf(\gamma)} &= e^{-it\left(\sum_{j=1}^4 f\left(\frac{r_j}{t}\right) - 3\right)} + e^{-it\left(\sum_{j=1}^4 f\left(\frac{r_j}{t}\right) - 3\right)} O\left(e^{i\frac{1}{t}\sum_{1 \leq i < j \leq 4} r_i r_j} - 1\right) \\ &= e^{i3t} \prod_{j=1}^4 e^{-i(t^2 - r_j^2)^{\frac{1}{2}}} + O\left(\frac{1}{t} \sum_{1 \leq i < j \leq 4} r_i r_j\right). \end{aligned}$$

Further, since  $\gamma \ll 1$ , we have

$$\frac{\chi\left(\frac{\gamma}{\sqrt{1-\gamma^2}}\right)}{(1-\gamma^2)^{\frac{1}{4}}} = 1 + O(\gamma) = 1 + O\left(\frac{1}{t} \sum_{j=1}^k r_j\right),$$

we see the contribution of  $\psi(0)$  to (46) is

$$\frac{(-2\pi i)^{\frac{1}{2}} e^{i3t}}{t^{\frac{1}{2}}} \prod_{i=1}^4 e^{-i(t^2-r_j^2)^{\frac{1}{2}}} \chi(r_j/t) + O\left(\frac{1}{t^{\frac{3}{2}}} \left[ \sum_{1 \leq i < j \leq 4} r_i r_j + \sum_{j=1}^4 r_j \right]\right).$$

This finishes the proof.  $\square$

We can now prove the main claim of this subsection.

*Proof of Proposition 3.5.* Using Lemma 3.6 we see that the contribution of  $I$  in (45) to the first integral in (42) is given by

$$(48) \quad -\frac{i(-2\pi i)^{\frac{1}{2}}}{\sqrt{mt^{\frac{1}{2}}}} \int_{\mathbb{R}^{12}} e^{3imt} \prod_{j=1}^4 \chi(r_j/t) e^{-im(t^2-r_j^2)^{\frac{1}{2}}} \frac{FV F v^* S_1 D_1 S_1 v F V F}{r_1 r_2 r_3 r_4} dy_1 dy_2 dx_1 dx_2 \\ + O\left(\frac{1}{t^{\frac{3}{2}}} \int_{\mathbb{R}^{12}} \left[ \sum_{1 \leq i < j \leq 4} r_i r_j + \sum_{j=1}^4 r_j + 1 \right] \left| \frac{FV F v^* S_1 D_1 S_1 v F V F}{r_1 r_2 r_3 r_4} \right| dy_1 dy_2 dx_1 dx_2 \right) \\ =: t^{-\frac{1}{2}} K_t(x, y) + O(t^{-\frac{3}{2}}).$$

The last inequality follows from the proof of Lemma 3.4 noting that

$$\frac{\sum_{1 \leq i < j \leq 4} r_i r_j + \sum_{j=1}^4 r_j + 1}{r_1 r_2 r_3 r_4} \lesssim \prod_{j=1}^4 (1 + r_j^{-1}).$$

Note that

$$K_t = C e^{3imt} \tilde{F}_t V \tilde{F}_t v^* S_1 D_1 S_1 v \tilde{F}_t V \tilde{F}_t,$$

where  $C = (-i)^{\frac{3}{2}} (2\pi)^{\frac{1}{2}} m^{-\frac{1}{2}}$  and  $\tilde{F}_t$  is an integral operator with kernel

$$\tilde{F}_t(x, y) = \frac{\chi(|x-y|/t) e^{-im[t^2-|x-y|^2]^{\frac{1}{2}}} F(x, y)}{|x-y|} = \chi(|x-y|/t) e^{-imt[1-(|x-y|/t)^2]^{\frac{1}{2}}} \mathcal{G}_0(x, y).$$

In particular, since  $S_1$  is of rank at most two,  $K_t$  is of rank at most two.

Note that since  $\frac{|x-y|}{t} \lesssim 1$ , we have

$$\chi(|x-y|/t) e^{-im[t^2-|x-y|^2]^{\frac{1}{2}}} = e^{-imt} + O(|x-y|^2/t) \\ = e^{-imt} + O\left(|x-y| \frac{|x-y|^j}{t^j}\right), \quad 0 \leq j \leq 1.$$

The last equality holds since  $\frac{|x-y|}{t} \lesssim 1$ . Using this we write

$$K_t(x, y) = C e^{-imt} [\mathcal{G}_0 V \mathcal{G}_0 v^* S_1 D_1 S_1 v \mathcal{G}_0 V \mathcal{G}_0](x, y) + \tilde{K}_t(x, y).$$



Since  $||[|x - y|^{1+j}\mathcal{G}_0(x, y)]|| \lesssim \langle x \rangle^j \langle y \rangle^j (1 + |x - y|^{-1})$ , we employ a similar argument as in Lemma 3.4 to show that  $|\tilde{K}_t(x, y)| \lesssim \langle x \rangle^j \langle y \rangle^j t^{-j}$ ,  $0 \leq j \leq 1$ .

By Corollary 4.4, we know that the rank of  $S_1$  is at most two. Hence, we can write

$$S_1 = \phi_1(x)\phi_1^*(y) + \phi_2(x)\phi_2^*(y)$$

where we pick  $\{\phi_1, \phi_2\}$  as the orthonormal basis of  $S_1 L^2$ . The self-adjointness of  $S_1 v \mathcal{G}_1 v^* S_1$  also allows us to pick the basis so that  $S_1 v \mathcal{G}_1 v^* S_1$  is diagonal in  $S_1 L^2$ , i.e.,

$$(49) \quad \langle M_{uc} v^* \phi_j, M_{uc} v^* \phi_i \rangle = \|M_{uc} v^* \phi_j\|_{\mathbb{C}^4}^2 \delta_{ij}, \quad i, j = 1, 2.$$

Using this one can show that

$$S_1 v \mathcal{G}_1 v^* S_1 = \frac{m}{2\pi} \begin{bmatrix} a^2 & 0 \\ 0 & b^2 \end{bmatrix} S_1, \quad \text{and} \quad D_1 = (S_1 v \mathcal{G}_1 v^* S_1)^{-1} = \frac{2\pi}{m} \begin{bmatrix} \frac{1}{a^2} & 0 \\ 0 & \frac{1}{b^2} \end{bmatrix} S_1,$$

where  $a = \|M_{uc} v^* \phi_1\|_{\mathbb{C}^4}$  and  $b = \|M_{uc} v^* \phi_2\|_{\mathbb{C}^4}$ . Therefore, the self-adjoint operator  $S_1 D_1 S_1$  can be rewritten as

$$[S_1 D_1 S_1](x, y) = \frac{2\pi}{ma^2} \phi_1(x) \phi_1^*(y) + \frac{2\pi}{mb^2} \phi_2(x) \phi_2^*(y).$$

Furthermore, Lemma 4.1 gives us that  $\phi_j = U v \psi_j$  for  $\psi_j = -\mathcal{G}_0 v^* \phi_j$  where  $(D_m + V - mI)\psi_j = 0$ . Using this (49) =  $\langle M_{uc} V \psi_i, M_{uc} V \psi_j \rangle$ . Noting that by definition of  $S_1$ , we have  $-S_1 = S_1 v \mathcal{G}_0 v^* U = U v \mathcal{G}_0 v^* S_1$ , we obtain

$$\begin{aligned} [\mathcal{G}_0 V \mathcal{G}_0 v^* S_1 D_1 S_1 v \mathcal{G}_0 V \mathcal{G}_0](x, y) &= [\mathcal{G}_0 v^* S_1 D_1 S_1 v \mathcal{G}_0](x, y) \\ &= \frac{2\pi}{ma^2} [\mathcal{G}_0 v^* \phi_1](x) [\mathcal{G}_0 v^* \phi_1]^*(y) + \frac{2\pi}{mb^2} [\mathcal{G}_0 v^* \phi_2](x) [\mathcal{G}_0 v^* \phi_2]^*(y) \\ &= \frac{2\pi}{ma^2} \psi_1(x) \psi_1^*(y) + \frac{2\pi}{mb^2} \psi_2(x) \psi_2^*(y) := \frac{m^{\frac{1}{2}}}{(-i)^{\frac{3}{2}} (2\pi)^{\frac{1}{2}}} P_r(x, y). \end{aligned}$$

Finally, note that if  $S_1$  is one dimensional it is generated by a single  $\phi(x)$  with  $\langle \phi, \phi \rangle = 1$ . In this case we obtain  $P_r(x, y) = \frac{(-2\pi i)^{\frac{3}{2}}}{m^{\frac{3}{2}} \|M_{uc} v^* \phi\|_{\mathbb{C}^4}^2} \psi(x) \psi^*(y)$ .

This finishes the proof of Proposition 3.5.  $\square$

We are now ready to prove Theorem 1.1.

*Proof of Theorem 1.1.* Using the Stone's formula, (7), and the expansion for the resolvent (19), we reduce our analysis to oscillatory integral bounds. Proposition 3.2 suffices to bound the contribution of the first three terms of (19) by  $\langle t \rangle^{-\frac{3}{2}}$  as an operator from  $L^1$  to  $L^\infty$ . The contribution of the final term in (19) is controlled by Proposition 3.5.  $\square$

**3.3. Dispersive estimate when there is a resonance of the second or third kind at the threshold.** In this section we will investigate dispersive estimate in the case when  $S_2 \neq 0$ . To establish the claim of Theorem 1.2, we devote this subsection to proving

**Proposition 3.8.** *Under the assumptions of Theorem 1.2, there is a finite rank operator  $K_t$  so that*

$$(50) \quad \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} [\mathcal{R}_0 V \mathcal{R}_0 v^* A^{-1} v \mathcal{R}_0 V \mathcal{R}_0](z)(x, y) dz = t^{-\frac{1}{2}} K_t(x, y) + O(t^{-\frac{3}{2}}),$$

where  $\sup_t \|K_t(x, y)\|_{L^1 \rightarrow L^\infty} < \infty$  and the error term is bounded uniformly in  $x, y$ . Moreover, the integral above is bounded uniformly in  $x, y, t$ .

*Proof.* Using the expansion for  $A^{-1}(z)$  from Lemma 2.14 in the integral above we consider

$$\frac{1}{z^2} \mathcal{R}_0 V \mathcal{R}_0 v^* S_2 D_3 S_2 v \mathcal{R}_0 V \mathcal{R}_0 + \frac{1}{z} \mathcal{R}_0 V \mathcal{R}_0 v^* \Omega v \mathcal{R}_0 V \mathcal{R}_0 + \mathcal{R}_0 V \mathcal{R}_0 v^* E v \mathcal{R}_0 V \mathcal{R}_0.$$

The last two terms can be handled similar to those we have already bounded in Subsection 3.2. In particular, the operator with decay rate  $t^{-\frac{1}{2}}$  is not necessarily rank at most two, but is finite rank. This is because instead of  $S_1 D_1 S_1$  here we have  $\Omega$ , which was shown to be finite rank in the proof of Lemma 2.14. Hence, it suffices to consider the integral

$$(51) \quad \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{\chi(z)}{z\sqrt{z^2+m^2}} [\mathcal{R}_0 V \mathcal{R}_0 v^* S_2 D_3 S_2 v \mathcal{R}_0 V \mathcal{R}_0](z)(x, y) dz.$$

Using the identity  $\mathcal{R}_0 = \mathcal{G}_0 + [\mathcal{R}_0 - \mathcal{G}_0]$ , and noting that  $S_2 v \mathcal{G}_1 = 0$  (see Corollary 4.3) we can rewrite

$$(52) \quad \begin{aligned} \mathcal{R}_0 V \mathcal{R}_0 v^* S_2 D_3 S_2 v \mathcal{R}_0 V \mathcal{R}_0 &= \mathcal{R}_0 V \mathcal{G}_0 v^* S_2 D_3 S_2 v \mathcal{G}_0 V \mathcal{R}_0 \\ &+ \mathcal{R}_0 V \mathcal{G}_0 v^* S_2 D_3 S_2 v [\mathcal{R}_0 - \mathcal{G}_0 - iz\mathcal{G}_1] V \mathcal{R}_0 + \mathcal{R}_0 V [\mathcal{R}_0 - \mathcal{G}_0 - iz\mathcal{G}_1] v^* S_2 D_3 S_2 v \mathcal{G}_0 V \mathcal{R}_0 \\ &+ \mathcal{R}_0 V [\mathcal{R}_0 - \mathcal{G}_0 - iz\mathcal{G}_1] v^* S_2 D_3 S_2 v [\mathcal{R}_0 - \mathcal{G}_0 - iz\mathcal{G}_1] V \mathcal{R}_0. \end{aligned}$$

Recall the expansions of (13) for  $\mathcal{R}_0$  given in Lemma 2.1, picking  $\ell = 0+$  we have

$$E_1(z)(x, y) := \frac{1}{z^2} [\mathcal{R}_0 - \mathcal{G}_0 - iz\mathcal{G}_1] = \mathcal{G}_2 - iz\mathcal{G}_3 + \tilde{O}_2(z^{1+}(r^{2+} + r^{-1})).$$

Therefore,  $E_1$  satisfies the hypothesis of Lemma 3.4 with  $\alpha = 2+$ . Hence using Lemma 3.4 with  $E_1$  as above,  $E = S_2 D_3 S_2$ , and  $E_2 = \mathcal{G}_0$  we see that the contribution of the second summand in (52) to (51) is  $O(\langle t \rangle^{-3/2})$  provided that  $\beta > 7$ . The contribution of third and fourth summands can be handled in the same manner.

Now we turn to the first term in the equation (52). By Lemma 4.6 below, we have the identity  $\mathcal{G}_0 v^* S_2 D_3 S_2 v \mathcal{G}_0 = \mathcal{G}_0 V \mathcal{G}_0 v^* S_2 D_3 S_2 v \mathcal{G}_0 V \mathcal{G}_0 = -2mP_m$ . Also note that, by (70), we have

$$\mathcal{G}_0 V P_m = P_m V \mathcal{G}_0 = -P_m.$$

Hence, the first term can be rewritten as

$$\begin{aligned}
(53) \quad & \frac{1}{2m} \mathcal{R}_0 V \mathcal{G}_0 v^* S_2 D_3 S_2 v \mathcal{G}_0 V \mathcal{R}_0 \\
& = -P_m + [\mathcal{R}_0 - \mathcal{G}_0] V P_m + P_m V [\mathcal{R}_0 - \mathcal{G}_0] - [\mathcal{R}_0 - \mathcal{G}_0] V P_m V [\mathcal{R}_0 - \mathcal{G}_0].
\end{aligned}$$

The contribution of  $P_m$  in (50) is zero since the integral is an odd principal value integral. Note that the contributions of the last three terms to the Stone's formula is bounded for small  $t$  by an argument similar to the proof of Lemma 3.4. The following lemma takes care of the contribution of the second and third terms to (52) when  $t > 1$ .

**Lemma 3.9.** *Under the assumptions of Theorem 1.2,*

$$(54) \quad K_1(t)(x, y) := \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{\chi(z)}{z\sqrt{z^2+m^2}} \left[ P_m V [\mathcal{R}_0 - \mathcal{G}_0] \right](z)(x, y) dz$$

is a finite rank operator and  $\|K_1(t)\|_{L^1 \rightarrow L^\infty} \lesssim t^{-\frac{1}{2}}$ .

*Proof.* First of all note that  $K_t$  is finite rank since  $P_m$  is independent of  $z$  and finite rank by Lemma 4.6. Therefore, it suffices to bound (54) by  $t^{-\frac{1}{2}}$  uniformly in  $x, y$ .

Using the definition of  $\mathcal{R}_0(\lambda)$ , and the equations (10), (15), we have

$$\begin{aligned}
(55) \quad & [\mathcal{R}_0 - \mathcal{G}_0](z)(y_1, y) = [-i\alpha \cdot \nabla + 2mI_{uc}] \left[ \frac{e^{iz|y-y_1|} - 1}{4\pi|y-y_1|} \right] + \frac{e^{iz|y-y_1|}}{4\pi|y-y_1|} [\sqrt{z^2+m^2} - m] \\
& = \left[ \frac{i\alpha \cdot (y-y_1)}{|y-y_1|^2} + 2mI_{uc} \right] \left[ \frac{e^{iz|y-y_1|} - 1}{4\pi|y-y_1|} \right] + \left[ z \frac{\alpha \cdot (y-y_1)}{|y-y_1|} + \tilde{O}_2(z^2) \right] \frac{e^{iz|y-y_1|}}{4\pi|y-y_1|}.
\end{aligned}$$

It is easy to see that the contribution of the last term is  $O(\langle t \rangle^{-\frac{1}{2}})$  by a single application of Lemma 3.1. Note that the uniform bound in  $x$  uses the boundedness of eigenfunctions, see Lemma 4.2.

Now we estimate the contribution of the first term in (55). Let

$$\begin{aligned}
(56) \quad & \tilde{K}_t(y, y_1) := \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{\chi(z)}{z\sqrt{z^2+m^2}} \left[ \frac{i\alpha \cdot (y-y_1)}{|y-y_1|^2} + 2mI_{uc} \right] \left[ \frac{e^{iz|y-y_1|} - 1}{4\pi|y-y_1|} \right] dz \\
& = i \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{\chi(z)}{\sqrt{z^2+m^2}} \left[ \frac{i\alpha \cdot (y-y_1)}{|y-y_1|^2} + 2mI_{uc} \right] \int_0^{|y-y_1|} \frac{e^{izb}}{4\pi|y-y_1|} db dz.
\end{aligned}$$

By Lemma 3.1 we have

$$\left| \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}+izb} \frac{\chi(z)}{\sqrt{z^2+m^2}} dz \right| \lesssim t^{-\frac{1}{2}},$$

uniformly in  $b$ . Therefore, by Fubini's theorem

$$|\tilde{K}_t(y, y_1)| \lesssim t^{-\frac{1}{2}} \int_0^{|y-y_1|} (|y-y_1|^{-2} + |y-y_1|^{-1}) db \lesssim t^{-\frac{1}{2}} (1 + |y-y_1|^{-1}).$$

Using the boundedness of eigenfunctions and the decay of  $V$ , we obtain

$$|[P_m V \tilde{K}_t](x, y)| \lesssim t^{-\frac{1}{2}},$$

uniformly in  $x, y$ , which finishes the proof.  $\square$

Now, we consider the contribution of the last term in (53) to (52) when  $t > 1$ .

**Lemma 3.10.** *Under the assumptions of Theorem 1.2 we have*

$$\sup_{x, y \in \mathbb{R}^3} \left| \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{\chi(z)}{z\sqrt{z^2+m^2}} [\mathcal{R}_0 - \mathcal{G}_0] V P_m V [\mathcal{R}_0 - \mathcal{G}_0] (z)(x, y) dz \right| \lesssim t^{-\frac{3}{2}}.$$

*Proof.* Using (55), we have

$$\begin{aligned} \frac{\mathcal{R}_0 - \mathcal{G}_0}{z} &= M + N, \quad \text{where} \\ M(z)(x, x_1) &:= [-i\alpha \cdot \nabla] \left[ \frac{e^{iz|x-x_1|} - 1}{4\pi z|x-x_1|} \right] + [\sqrt{z^2+m^2} - m] \frac{e^{iz|x-x_1|}}{4\pi z|x-x_1|}, \\ N(z, |x-x_1|) &:= 2mI_{uc} \left[ \frac{e^{iz|x-x_1|} - 1}{4\pi z|x-x_1|} \right]. \end{aligned}$$

We will see that the operator  $M$  satisfies suitable bounds. However, the operator  $N$  does not, instead we need to use that  $\mathcal{G}_1 V P_m = \frac{im}{2\pi} M_{uc} V P_m = 0$  and  $P_m V \mathcal{G}_1 = \frac{im}{2\pi} P_m V M_{uc} = 0$ . Hence, we can replace the term  $N(z, |x-x_1|)$  with

$$\mathcal{N}(z)(x, x_1) = N(z, |x-x_1|) - N(z, \langle x \rangle)$$

on both sides of  $V P_m V$ .

The following lemma contains the required bounds:

**Lemma 3.11.** *We have*

$$(57) \quad M(z)(x, x_1) + \mathcal{N}(z)(x, x_1) = [\langle x_1 \rangle + |x-x_1|^{-1}] O(z),$$

$$(58) \quad \partial_z (M(z) + \mathcal{N}(z))(x, x_1) = e^{iz|x-x_1|} [\langle x_1 \rangle + |x-x_1|^{-1}] \tilde{O}_1(1) + \langle x_1 \rangle \tilde{O}_1(1).$$

The proof of this lemma is given below. We finish the proof of Lemma 3.10 using Lemma 3.11.

We start with applying integration by parts to the integral

$$(59) \quad \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \frac{z\chi(z)}{\sqrt{z^2+m^2}} (M(z) + \mathcal{N}(z))(x, x_1) (M(z) + \mathcal{N}(z))(y, y_1) dz.$$

We only consider the case when the derivative falls on  $(M(z) + \mathcal{N}(z))(y, y_1)$ . The other cases are similar. We therefore consider

$$t^{-1} \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \chi(z) (M(z) + \mathcal{N}(z))(x, x_1) \partial_z (M(z) + \mathcal{N}(z))(y, y_1) dz.$$

Using Lemma 3.11, we can write this integral as

$$\begin{aligned} &t^{-1} (\langle x_1 \rangle + |x-x_1|^{-1}) (\langle y_1 \rangle + |y-y_1|^{-1}) \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}+iz|y-y_1|} \chi(z) \tilde{O}_1(z) dz \\ &+ t^{-1} (\langle x_1 \rangle + |x-x_1|^{-1}) \langle y_1 \rangle \int_{-\infty}^{\infty} e^{-it\sqrt{z^2+m^2}} \chi(z) \tilde{O}_1(z) dz. \end{aligned}$$

Applying Lemma 3.1 with the phase  $\phi(z) = -t\sqrt{z^2 + m^2} + z|y - y_1|$  and the phase  $\phi(z) = -t\sqrt{z^2 + m^2}$  in each integral respectively, yields the bound

$$t^{-\frac{3}{2}}(\langle x_1 \rangle + |x - x_1|^{-1})(\langle y_1 \rangle + |y - y_1|^{-1}).$$

This establishes the claim since  $(\langle x_1 \rangle + |x - x_1|^{-1})|V(x_1)| \in L^2_{x_1}$  uniformly in  $x$ .  $\square$

*Proof of Lemma 3.11.* We start with  $M$ . It is clear that second summand in the definition of  $M$  satisfies (57) and (58). Let  $p = |x - x_1|$ . The first summand in the definition of  $M$  is (omitting the factors of  $\frac{1}{4\pi}$ )

$$(60) \quad [-i\alpha \cdot \nabla] \left[ \frac{e^{iz|x-x_1|} - 1}{z|x-x_1|} \right] = \frac{i\alpha \cdot (x - x_1)}{p^2} \left[ \frac{e^{izp} - 1}{zp} \right] + \frac{\alpha \cdot (x - x_1)e^{izp}}{p^2} = O(p^{-1}).$$

Note that if  $|z|p \gtrsim 1$ , this term is bounded by  $|z|$ . If  $|z|p \lesssim 1$ , then we have  $e^{izp} = 1 + izp - \frac{1}{2}z^2p^2 + \tilde{O}_2(z^3p^3)$ . Plugging this into (60), we obtain

$$(61) \quad [-i\alpha \cdot \nabla] \left[ \frac{e^{iz|x-x_1|} - 1}{z|x-x_1|} \right] = [-i\alpha \cdot \nabla] \left[ i - \frac{zp}{2} + \tilde{O}_2(z^2p^2) \right] = O(z).$$

Taking the  $z$  derivative of the first summand, we obtain

$$\begin{aligned} \partial_z [-i\alpha \cdot \nabla] \left[ \frac{e^{iz|x-x_1|} - 1}{z|x-x_1|} \right] &= \frac{i\alpha \cdot (x - x_1)}{p^2} \left[ \frac{e^{izp}ip}{zp} - \frac{e^{izp} - 1}{z^2p} \right] + \frac{i\alpha \cdot (x - x_1)e^{izp}}{p} \\ &= e^{izp} \frac{i\alpha \cdot (x - x_1)}{p} \left[ \frac{e^{-izp} - 1 + izp}{z^2p^2} + 1 \right] = e^{izp} \tilde{O}_1(1). \end{aligned}$$

The last bound follows by noting that the numerator is  $\tilde{O}_1(z^2p^2)$  by a Taylor expansion when  $zp \lesssim 1$ .

To obtain the bounds for  $\mathcal{N}$ , consider the function

$$f(r) = \frac{e^{izr} - 1}{zr}.$$

Then, ignoring the constant factors,  $\mathcal{N}(z) = f(p) - f(q)$ , where  $q := \langle x \rangle$ . Using the mean value theorem we have

$$(62) \quad |\mathcal{N}(z)(x, x_1)| \lesssim \frac{|p - q|}{|z|} \max_r \left| \frac{ize^{izr} - e^{izr} + 1}{r^2} \right| \lesssim |z| \langle x_1 \rangle.$$

In the last inequality we used the fact that  $(ize^{izr} - e^{izr} + 1) = \tilde{O}_1(z^2r^2)$ .

Further,

$$\begin{aligned} \partial_z \mathcal{N}(z)(x, x_1) &= \frac{izpe^{izp} - e^{izp} + 1}{z^2p} - \frac{izqe^{izq} - e^{izq} + 1}{z^2q} \\ &= ie^{izp} \frac{1 - e^{iz(q-p)}}{z} - \frac{\mathcal{N}(z)}{z}. \end{aligned}$$

By the mean value theorem we have

$$\frac{1 - e^{iz(q-p)}}{z} = \langle x_1 \rangle \tilde{O}_1(1).$$

Finally, note that the inequality (62) and the calculation

$$\left| \partial_z \frac{\mathcal{N}(z)}{z} \right| = \left| -\frac{1}{z^2} \mathcal{N}(z) + \frac{\partial_z \mathcal{N}(z)}{z} \right| \lesssim \frac{\langle x_1 \rangle}{|z|}$$

imply that

$$\frac{\mathcal{N}(z)}{z} = \langle x_1 \rangle \tilde{O}_1(1).$$

□

This completes the proof of Proposition 3.8. □

*Proof of Theorem 1.2.* The proof follows as in the proof of Theorem 1.1 using Proposition 3.8 instead of Proposition 3.5. □

**Remark 3.12.** *This method also applies to the analysis of the Schrödinger operator considered in [20] and [40]. In particular, it implies that the  $t^{-\frac{1}{2}}$  term is a time dependent finite rank operator when zero is not a regular point of the spectrum. This gives an alternative proof to Yajima's theorem in [40]. In [20], such a result was obtained only in the case when there is a resonance of the first kind.*

#### 4. CLASSIFICATION OF THRESHOLD SPECTRAL SUBSPACES

**Lemma 4.1.** *Assume  $|v(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-}$ . Then  $\phi \in S_1 L^2(\mathbb{R}^3) \setminus \{0\}$  if and only if  $\phi = Uv\psi$  for some  $\psi \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3) \setminus \{0\}$  which is a distributional solution of  $(D_m + V - mI)\psi = 0$ . Furthermore,  $\psi = -\mathcal{G}_0 v^* \phi$  and  $\psi$  is a bounded function.*

*Proof.* If  $\phi \in S_1 L^2(\mathbb{R}^3) \setminus \{0\}$ , then by Definition 2.8,  $(U + v\mathcal{G}_0 v^*)\phi = 0$ . Since  $U^2 = I$ ,

$$(63) \quad \phi = -Uv\mathcal{G}_0 v^* \phi = Uv\psi, \quad \text{where } \psi := -\mathcal{G}_0 v^* \phi.$$

Using (15) and (5) with  $\lambda = m$ , we obtain

$$(64) \quad (D_m - mI)\psi = -(D_m - mI)\mathcal{G}_0 v^* \phi = -(D_m - mI)(D_m + mI)G_0 v^* \phi \\ = \Delta G_0 v^* \phi = -v^* \phi = -v^* Uv\psi = -V\psi.$$

Therefore,  $(D_m + V - mI)\psi = 0$ . In the fourth equality above, we used the fact that  $\Delta G_0 v^* \phi = -v^* \phi$  holds since  $v^* \phi \in L^{2, \frac{3}{2}+}$ , see Lemma 2.4 in [26].

Now we prove that  $\psi \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3)$ . Note that

$$(65) \quad \psi = -[-i\alpha \cdot \nabla + 2mI_{uc}]G_0 v^* \phi = -[-i\alpha \cdot \nabla + 2mI_{uc}]\frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{v^*(y)\phi(y)}{|x-y|} \\ = \frac{1}{4\pi} \int_{\mathbb{R}^3} i\alpha \cdot (x-y) \frac{v^*(y)\phi(y)}{|x-y|^3} - 2mI_{uc} \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{v^*(y)\phi(y)}{|x-y|} =: \psi_1 + \psi_2.$$

Since the integrals in equation can be bounded by fractional integral operators, we can use Lemma 2.3 in [25]. We have  $\psi_1 \in L^2(\mathbb{R}^3) \subseteq L^{2, -\frac{1}{2}-}(\mathbb{R}^3)$  provided  $|v(x)| \lesssim \langle x \rangle^{-1}$ ; and  $\psi_2 \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3)$  provided  $|v(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-}$ .

Conversely, assume that  $\phi = Uv\psi$  for some  $\psi \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3)$  satisfying  $(D_m + V - mI)\psi = 0$ . Then  $\phi \in L^{2, 1+}$ , and by a calculation similar to (64), we have

$$(D_m - mI)\psi = -V\psi = -v^*\phi = \Delta G_0 v^*\phi = -(D_m - mI)(D_m + mI)G_0 v^*\phi = -(D_m - mI)\mathcal{G}_0 v^*\phi.$$

Thus, also using (5) with  $\lambda = -m$ , we have

$$\Delta(\psi + \mathcal{G}_0 v^*\phi) = (D_m + mI)(D_m - mI)(\psi + \mathcal{G}_0 v^*\phi) = 0.$$

Noting that  $\psi + \mathcal{G}_0 v^*\phi \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3)$ , we conclude that (see [26])  $\psi + \mathcal{G}_0 v^*\phi = 0$ . Notice that this also implies that the free Dirac has no threshold resonances. Therefore,

$$(U + v\mathcal{G}_0 v^*)\phi = v\psi + v\mathcal{G}_0 v^*\phi = 0,$$

and hence  $\phi \in S_1 L^2$ .

Since  $\phi = Uv\psi$ , if  $\phi \neq 0$ , then  $\psi \neq 0$ . The reverse implication follows from  $\psi = -\mathcal{G}_0 v^*\phi$ .

Finally, using (63) we have  $\psi = -\mathcal{G}_0 V\psi$ . Iterating this identity we obtain  $\psi = \mathcal{G}_0 V\mathcal{G}_0 V\psi$ . Therefore, by a calculation identical to the one in Remark 2.4, we see that  $\psi$  bounded.  $\square$

**Lemma 4.2.** *Suppose  $|v(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-}$ . Fix  $\phi = Uv\psi \in S_1 L^2$ , where  $\psi \in L^{2, -\frac{1}{2}-}(\mathbb{R}^3) \setminus \{0\}$  is a distributional solution of  $(D_m + V - mI)\psi = 0$ . Then  $\phi \in S_2 L^2(\mathbb{R}^3)$  if and only if  $\psi \in L^2(\mathbb{R}^3)$ . Moreover, any threshold eigenfunction,  $\psi$ , is a bounded function.*

*Proof.* The boundedness of  $\psi$  and the equality  $(D_m + V - mI)\psi = 0$  were obtained in the previous lemma. First note that if  $\phi \in S_2 L^2(\mathbb{R}^3)$ , namely  $S_1 v\mathcal{G}_1 v^*\phi = 0$ , then

$$0 = \langle vM_{uc}v^*\phi, \phi \rangle = \langle M_{uc}v^*\phi, M_{uc}v^*\phi \rangle_{\mathbb{C}^4} = \|M_{uc}v^*\phi\|_{\mathbb{C}^4}^2,$$

Hence,  $M_{uc}v^*\phi = 0$ . It is also clear that if  $M_{uc}v^*\phi = 0$ , then  $\phi \in S_2 L^2$ .

Also note that in the proof of Lemma 4.1, we showed that  $\psi = \psi_1 + \psi_2$  and  $\psi_1 \in L^2(\mathbb{R}^3)$ . Therefore it suffices to prove that  $M_{uc}v^*\phi = 0$  if and only if  $\psi_2 \in L^2$ . Recalling (65) we can write  $\psi_2$  as

$$(66) \quad \psi_2(x) = \frac{m}{2\pi} I_{uc} \int_{\mathbb{R}^3} v^*(y)\phi(y) \left[ \frac{1}{|x-y|} - \frac{1}{\langle x \rangle} \right] dy + \frac{m}{2\pi \langle x \rangle} [M_{uc}v^*\phi].$$

Using [20, Lemma 6] we see that the first integral above is in  $L^2(\mathbb{R}^3)$ . Since  $\frac{1}{\langle x \rangle} \notin L^2(\mathbb{R}^3)$ , we conclude that  $\psi_2 \in L^2$  if and only if  $M_{uc}v^*\phi = 0$ .  $\square$

A useful consequence of this proof is the following orthogonality condition and the fact that the rank of  $S_1 - S_2$  is at most two.

**Corollary 4.3.** *We have the identities*

$$S_2 v \mathcal{G}_1 = \mathcal{G}_1 v^* S_2 = \frac{m}{2\pi} M_{uc} v^* S_2 = \frac{m}{2\pi} S_2 v M_{uc} = 0.$$

**Corollary 4.4.** *Assume  $|v(x)| \lesssim \langle x \rangle^{-\frac{3}{2}-}$ . Then the rank of  $S_1 - S_2$  is at most two.*

*Proof.* We consider the representation in (65). We have already shown that  $\psi_1 \in L^2$ . By (66) and the discussion following it, we can write  $\psi_2$  as

$$\frac{m}{2\pi \langle x \rangle} [M_{uc} v^* \phi] + O_{L^2}(1) = \frac{1}{\langle x \rangle} (a_1, a_2, 0, 0)^T + O_{L^2}(1).$$

The constants  $a_j = \frac{m}{2\pi} \int_{\mathbb{R}^3} [v^*(y) \phi(y)]_j dy$  are finite by the assumed decay of  $v^*$ .  $\square$

**Lemma 4.5.** *Assume  $|v(x)| \lesssim \langle x \rangle^{-\frac{5}{2}-}$ . Then  $S_2 v \mathcal{G}_2 v^* S_2$  is invertible as an operator in  $S_2 L^2(\mathbb{R}^3)$ .*

*Proof.* Since  $S_2 v \mathcal{G}_2 v^* S_2$  is a compact operator it is enough to show that its kernel is empty. Assume that for some  $\phi \in S_2 L^2(\mathbb{R}^3)$ ,  $S_2 v \mathcal{G}_2 v^* S_2 \phi = 0$ , i.e.,  $\langle \mathcal{G}_2 v^* \phi, v^* \phi \rangle = 0$ . By Corollary 4.3  $\mathcal{G}_1 v^* \phi = 0$ . Using these equalities in (13), under the decay condition on  $|v(x)|$

$$(67) \quad 0 = \langle \mathcal{G}_2 v^* \phi, v^* \phi \rangle = - \lim_{z \rightarrow 0} \frac{1}{z^2} \langle [\mathcal{R}_0(\lambda) - \mathcal{G}_0] v^* \phi, v^* \phi \rangle$$

where  $\lambda = \sqrt{z^2 + m^2}$ . The following equality holds for  $z = i\omega$  and  $0 < \omega \ll m$ ,

$$\frac{1}{z^2} \langle [\mathcal{R}_0(\lambda) - \mathcal{G}_0] v^* \phi, v^* \phi \rangle = \int_{\mathbb{R}^3} \langle K(\omega, \xi) \widehat{v^* \phi(\xi)}, \widehat{v^* \phi(\xi)} \rangle_{\mathbb{C}^4} d\xi,$$

where

$$K(\omega, \xi) = \frac{1}{\omega^2 |\xi|^2} \begin{pmatrix} 2m & 0 & \xi_3 & \bar{\eta} \\ 0 & 2m & \eta & -\xi_3 \\ \xi_3 & \bar{\eta} & 0 & 0 \\ \eta & -\xi_3 & 0 & 0 \end{pmatrix} - \frac{1}{\omega^2 (\omega^2 + |\xi|^2)} \begin{pmatrix} m + \sqrt{m^2 - \omega^2} & 0 & \xi_3 & \bar{\eta} \\ 0 & m + \sqrt{m^2 - \omega^2} & \eta & -\xi_3 \\ \xi_3 & \bar{\eta} & \sqrt{m^2 - \omega^2} - m & 0 \\ \eta & -\xi_3 & 0 & \sqrt{m^2 - \omega^2} - m \end{pmatrix}.$$

Here  $\xi = (\xi_1, \xi_2, \xi_3)$  and  $\eta = \xi_2 + i\xi_1$ .

Let  $\tau := \frac{|\xi|^2}{\omega^2} (m - \sqrt{m^2 - \omega^2})$ , then  $K(\omega, \xi)$  can be written as

$$(68) \quad \frac{1}{|\xi|^2 (\omega^2 + |\xi|^2)} \begin{pmatrix} 2m + \tau & 0 & \xi_3 & \eta \\ 0 & 2m + \tau & \eta & -\xi_3 \\ \xi_3 & \bar{\eta} & \tau & 0 \\ \eta & -\xi_3 & 0 & \tau \end{pmatrix}.$$



The eigenvalues of  $K(\omega, \xi)$  are

$$\lambda_{1,2} = \frac{m + \tau + \sqrt{m^2 + |\xi|^2}}{|\xi|^2(\omega^2 + |\xi|^2)}, \quad \lambda_{3,4} = \frac{m + \tau - \sqrt{m^2 + |\xi|^2}}{|\xi|^2(\omega^2 + |\xi|^2)}.$$

Note that the eigenvalues are real and for any  $\xi \neq 0$  they are positive. Hence,  $K(\omega, \xi)$  is self-adjoint and positive definite for any  $\xi \neq 0$ . One can also check that the eigenvalues are nonincreasing functions of  $\omega \in (0, m)$ . Hence, we can use monotone convergence theorem and take the limit into the integral (67) to obtain

$$0 = \lim_{\omega \rightarrow 0^+} \int_{\mathbb{R}^3} \langle K(\omega, \xi) v^* \hat{\phi}(\xi), v^* \hat{\phi}(\xi) \rangle_{\mathbb{C}^4} d\xi = \int_{\mathbb{R}^3} \langle K(0, \xi) v^* \hat{\phi}(\xi), v^* \hat{\phi}(\xi) \rangle_{\mathbb{C}^4} d\xi$$

where

$$K(0, \xi) = \frac{1}{|\xi|^4} \begin{pmatrix} 2m + \frac{|\xi|^2}{2m} & 0 & \xi_3 & \bar{\eta} \\ 0 & 2m + \frac{|\xi|^2}{2m} & \eta & -\xi_3 \\ \xi_3 & \bar{\eta} & \frac{|\xi|^2}{2m} & 0 \\ \eta & -\xi_3 & 0 & \frac{|\xi|^2}{2m} \end{pmatrix}.$$

Note that this matrix is also self-adjoint and positive definite. Therefore,  $v^* \hat{\phi}(\xi) = 0$ . Since  $v^* \phi(\xi)$  has  $L^1$  entries,  $v^* \phi = 0$ . Recall that the fact that  $\phi \in S_1 L^2(\mathbb{R}^3)$  implies that  $\phi = U v^* \psi$  for  $\psi = -\mathcal{G}_0 v^* \phi$ . Hence, we conclude that  $\phi = 0$ .  $\square$

In addition, one can see that

$$K(0, \xi) = \frac{1}{2m} \frac{1}{|\xi|^4} \begin{pmatrix} 2m & 0 & \xi_3 & \bar{\eta} \\ 0 & 2m & \eta & -\xi_3 \\ \xi_3 & \bar{\eta} & 0 & 0 \\ \eta & -\xi_3 & 0 & 0 \end{pmatrix}^2.$$

Therefore, for any  $\phi \in S_2 L^2$  we have

$$(69) \quad \langle \mathcal{G}_2 v^* \phi, v^* \phi \rangle = -\frac{1}{2m} \langle \mathcal{G}_0 v^* \phi, \mathcal{G}_0 v^* \phi \rangle.$$

**Lemma 4.6.** *The operator  $P_m = -\frac{1}{2m} \mathcal{G}_0 V \mathcal{G}_0 v^* S_2 D_3 S_2 v \mathcal{G}_0 V \mathcal{G}_0$  equals the finite rank, orthogonal projection in  $L^2(\mathbb{R}^3)$  onto the eigenspace of  $H = D_m + V$  at threshold  $m$ .*

*Proof.* First recall that  $S_2 \leq S_1$  is finite dimensional. Using (63) we have  $S_2 = -S_2 v \mathcal{G}_0 v^* U$  and consequently

$$(70) \quad S_2 v \mathcal{G}_0 V \mathcal{G}_0 = S_2 v \mathcal{G}_0 v^* U v \mathcal{G}_0 = -S_2 v \mathcal{G}_0.$$

Similarly,  $\mathcal{G}_0 V \mathcal{G}_0 v^* S_2 = -\mathcal{G}_0 v^* S_2$ . Therefore,  $P_m = -\frac{1}{2m} \mathcal{G}_0 v^* S_2 D_3 S_2 v \mathcal{G}_0$ .

Let  $\{\phi_j\}_{j=1}^N$  be an orthonormal basis for the  $S_2 L^2(\mathbb{R}^3)$ , the range of  $S_2$ . Then, by Lemma 4.1, we have

$$(71) \quad \phi_j = U v \psi_j, \quad \psi_j = -\mathcal{G}_0 v^* \phi_j, \quad 1 \leq j \leq N,$$

where  $\psi_j \in L^2$ ,  $j = 1, 2, \dots, N$ , are eigenvectors. This implies that the range of  $P_m$  is contained in the span of  $\{\psi_j\}_{j=1}^N$ .

Since  $\{\phi_j\}_{j=1}^N$  is linearly independent, we have that  $\{\psi_j\}_{j=1}^N$  is linearly independent, and hence it is a basis for  $m$  energy eigenspace. Using the orthonormal basis for  $S_2 L^2(\mathbb{R}^3)$ , we have that for any  $f \in L^2$ ,  $S_2 f = \sum_{j=1}^N \langle f, \phi_j \rangle \phi_j$ . Therefore, we have

$$(72) \quad S_2 v \mathcal{G}_0 f = \sum_{j=1}^N \langle f, \mathcal{G}_0 v^* \phi_j \rangle \phi_j = - \sum_{j=1}^N \langle f, \psi_j \rangle \phi_j.$$

We claim that, for each  $i_0, j_0 \in \{1, 2, \dots, N\}$ ,

$$\langle \psi_{i_0}, P_m \psi_{j_0} \rangle = \langle \psi_{i_0}, \psi_{j_0} \rangle.$$

This implies the range of  $P_m$  is equal to the span of  $\{\psi_j\}_{j=1}^N$  and that  $P_m$  is the identity operator in the range of  $P_m$ . Since  $P_m$  is self-adjoint the assertion of the lemma holds.

Recall  $D_3 := (S_2 v \mathcal{G}_2 v^* S_2)^{-1}$ . Let  $A = \{A_{ij}\}_{i,j=1}^N$ ,  $B = \{B_{ij}\}_{i,j=1}^N$  be the matrix representations of  $S_2 v \mathcal{G}_2 v^* S_2$  and  $D_3$  with respect to the orthonormal basis  $\{\phi_j\}_{j=1}^N$  of  $S_2$ . Using (69) and polarization,

$$\begin{aligned} A_{ij} &= \langle \phi_j, S_2 v \mathcal{G}_2 v^* S_2 \phi_i \rangle = -\frac{1}{2m} \langle \mathcal{G}_0 v^* \phi_j, \mathcal{G}_0 v^* \phi_i \rangle = -\frac{1}{2m} \langle \psi_j, \psi_i \rangle, \\ B_{ij} &= A_{ij}^{-1} = \langle \phi_j, D_3 \phi_i \rangle. \end{aligned}$$

Using this and (72), we have

$$\begin{aligned} \langle \psi_{i_0}, P_m \psi_{j_0} \rangle &= -\frac{1}{2m} \langle S_2 v \mathcal{G}_0 \psi_{i_0}, D_3 S_2 v \mathcal{G}_0 \psi_{j_0} \rangle \\ &= -\frac{1}{2m} \left\langle \sum_{i=1}^N \langle \psi_{i_0}, \psi_i \rangle \phi_i, D_3 \sum_{j=1}^N \langle \psi_{j_0}, \psi_j \rangle \phi_j \right\rangle = -\frac{1}{2m} \sum_{i,j=1}^N \langle \psi_{i_0}, \psi_i \rangle \langle \psi_j, \psi_{j_0} \rangle \langle \phi_i, D_3 \phi_j \rangle \\ &= -2m \sum_{i,j=1}^N A_{i,i_0} B_{j,i} A_{j_0,j} = -2m A_{j_0,i_0} = \langle \psi_{i_0}, \psi_{j_0} \rangle. \end{aligned}$$

This finishes the proof of the claim and the lemma.  $\square$

## REFERENCES

- [1] M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards Applied Mathematics Series, 55. For sale by the Superintendent of Documents, U.S. Government Printing Office, Washington, D.C. 1964
- [2] S. Agmon, *Spectral properties of Schrödinger operators and scattering theory*. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), no. 2, 151–218.
- [3] E. Balslev E. and B. Helffer, *Limiting absorption principle and resonances for the Dirac operator*, Advances in Advanced Mathematics 13 (1992), 186–215.
- [4] M. Beceanu, *Dispersive estimates in  $\mathbb{R}^3$  with threshold eigenstates and resonances*. Anal. PDE 9 (2016), no. 4, 813–858.

- [5] I. Bejenaru and S. Herr, *The cubic Dirac equation: small initial data in  $H^{1/2}(\mathbb{R}^3)$* . Commun. Math. Phys. 335 (2015), 43–82.
- [6] ———, *The cubic Dirac equation: small initial data in  $H^{1/2}(\mathbb{R}^2)$* . Commun. Math. Phys. 343 (2016), 515–562.
- [7] A. Berthier and V. Georgescu, *On the point spectrum of Dirac operators*. J. Funct. Anal. 71 (1987), no. 2, 309–338.
- [8] N. Boussaid, *Stable directions for small nonlinear Dirac standing waves*. Comm. Math. Phys. 268 (2006), no. 3, 757–817.
- [9] N. Boussaid, N., P. D’Ancona and L. Fanelli, *Virial identity and weak dispersion for the magnetic Dirac equation*. J. Math. Pures Appl., 95:137–150, 2011.
- [10] N. Boussaid and S. Golenia, *Limiting absorption principle for some long range perturbations of Dirac systems at threshold energies*. Comm. Math. Phys. 299 (2010), no. 3, 677–708.
- [11] F. Cacciafesta, *Virial identity and dispersive estimates for the  $n$ -dimensional Dirac equation*, J. Math. Sci. Univ. Tokyo 18 (2011), 1–23.
- [12] F. Cacciafesta and E. Seré, *Local smoothing estimates for the massless Dirac equation in 2 and 3 dimensions*. J. Funct. Anal. 271 (2016) no.8, 2339–2358.
- [13] P. A. Cojuhari, *On the finiteness of the discrete spectrum of the Dirac operator*. Rep. Math. Phys. 57 (2006), no. 3, 333–341.
- [14] A. Comech, T. Phan and A. Stefanov, *Asymptotic stability of solitary waves in generalized Gross-Neveu model*. To appear in Ann. Inst. H. Poincaré Anal. Non Linéaire. arXiv:1407.0606
- [15] P. D’Ancona and L. Fanelli, *Decay estimates for the wave and Dirac equations with a magnetic potential*. Comm. Pure Appl. Math. 60 (2007), no. 3, 357–392.
- [16] M. B. Erdoğan and W. R. Green, *Dispersive estimates for the Schrödinger equation for  $C^{\frac{n-3}{2}}$  potentials in odd dimensions*. Int. Math. Res. Notices 2010:13, 2532–2565.
- [17] ———, *Dispersive estimates for Schrödinger operators in dimension two with obstructions at zero energy*. Trans. Amer. Math. Soc. 365 (2013), 6403–6440.
- [18] ———, *A weighted dispersive estimate for Schrödinger operators in dimension two*. Commun. Math. Phys. 319 (2013), 791–811.
- [19] ———, *The Dirac equation in two dimensions: Dispersive estimates and classification of threshold obstructions*. Preprint, 2016. <http://arxiv.org/abs/1606.00871>
- [20] M. B. Erdoğan and W. Schlag, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or an eigenvalue at zero energy in dimension three: I*, Dynamics of PDE 1 (2004), 359–379.
- [21] ———, *Dispersive estimates for Schrödinger operators in the presence of a resonance and/or eigenvalue at zero energy in dimension three: II*. J. Anal. Math. 99 (2006), 199–248.
- [22] M. Escobedo and L. Vega, *A semilinear Dirac equation in  $H^s(\mathbb{R}^3)$  for  $s > 1$* . SIAM J. Math. Anal. 28 (1997), no. 2, 338–362.
- [23] Georgescu, V., and Măntoiu, M. *On the spectral theory of singular Dirac type Hamiltonians*. J. Operator Theory 46 (2001), no. 2, 289–321.
- [24] M. Goldberg and W. Schlag, *Dispersive estimates for Schrödinger operators in dimensions one and three*. Comm. Math. Phys. vol. 251, no. 1 (2004), 157–178.
- [25] A. Jensen, *Spectral properties of Schrödinger operators and time-decay of the wave functions results in  $L^2(\mathbb{R}^m)$ ,  $m \geq 5$* . Duke Math. J. 47 (1980), no. 1, 57–80.

- [26] A. Jensen and T. Kato. *Spectral properties of Schrödinger operators and time-decay of the wave functions*. Duke Math. J. 46 (1979), no. 3, 583–611.
- [27] A. Jensen and G. Nenciu. *A unified approach to resolvent expansions at thresholds*. Rev. Mat. Phys. vol. 13, no. 6 (2001), 717–754.
- [28] E. Kopylova *Dispersion estimates for 2D Dirac equation*. Asymptot. Anal. 84 (2013), no. 1–2, 35–46.
- [29] J. Krieger and W. Schlag, *On the focusing critical semi-linear wave equation*. Amer. J. Math. 129 (2007), no. 3, 843–913.
- [30] Kurbenin, O. I. *The discrete spectra of the Dirac and Pauli operators*. Topics in Mathematical Physics, Vol. 3, Spectral Theory, p.43–52, M.S. Birman ed., Consultants Bureau, 1969.
- [31] A. Masaharu and O. Yamada, *Essential selfadjointness and invariance of the essential spectrum for Dirac operators*. Publ. Res. Inst. Math. Sci. 18 (1982), no. 3, 973–985.
- [32] M. Murata, *Asymptotic expansions in time for solutions of Schrödinger-type equations* J. Funct. Anal. 49 (1) (1982), 10–56.
- [33] M. Reed and B. Simon. *Methods of Modern Mathematical Physics I: Functional Analysis, IV: Analysis of Operators*, Academic Press, New York, NY, 1972.
- [34] S. N. Roze, *On the spectrum of the Dirac operator* Theoret. and Math. Phys. 2 (1970), no. 3, 377–382
- [35] W. Schlag, *Dispersive estimates for Schrödinger operators in dimension two*. Comm. Math. Phys. 257 (2005), no. 1, 87–117.
- [36] E. Stein, *Harmonic analysis real-variable methods, orthogonality, and oscillatory integrals*. Princeton Univ. Press, Princeton, NJ, 1993.
- [37] B. Thaller, *The Dirac equation*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1992.
- [38] M. Thompson, *The absence of embedded eigenvalues in the continuous spectrum for perturbed Dirac operators*. Boll. Un. Mat. Ital. A (5) 13 (1976), no. 3, 576–585.
- [39] V. Vogelsang, *Absence of embedded eigenvalues of the Dirac equation for long range potentials*. Analysis 7 (1987), no. 3–4, 259–274.
- [40] K. Yajima, *Dispersive estimate for Schrödinger equations with threshold resonance and eigenvalue*, Comm. Math. Phys. 259 (2005), 475–509.
- [41] O. Yamada, *A remark on the limiting absorption method for Dirac operators*. Proc. Japan Acad. Ser. A Math. Sci. 69 (1993), no. 7, 243–246.

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